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Stochastic Transport: Basic Physics for ITER

Why do we need to investigate „anomalous“ transport?
„Fluctuation“ induced transport

„Manageable“ stochasticity: ergodic divertors
Experimental signatures: ELM mitigation, toroidal spin-up, heat flux patterns, runaways
Symplectic field line and drift mappings
Characterization of incomplete chaos
Transport along tangles: Heat flux patterns

Estimates from statistical plasma physics
Ab initio stochastic transport theory for small and large Kubo numbers
Need for further basic physics research

There is confidence, based on wide-ranging experience in existing fusion devices, that the fusion performance of ITER will meet the reference target. However, several key operational aspects remain the subject of focused R&D activities which aim to ensure the reliable operation of the device at or beyond its design capability. … To develop plasma scenarios in which high fusion power production is combined with high confinement and plasma pressure, control of heat and particle fluxes, … the fusion community will need to confront a range of challenges involving plasma physics understanding …

[D.J. Campbell, this meeting]

Performance projections to ITER rely on the H-mode global energy confinement time scaling, since models of local transport … are not yet considered to be sufficiently accurate to replace scaling-based extrapolations.

[D.J. Campbell, Phys. Plasmas 8, 2041 (2001)]

Thermal energy confinement time (s) : 

\[ \tau_{E,\text{th}} = 0.0562 \ H \ I_p^{0.93} \ B_T^{0.15} \ P^{-0.69} \ n_e^{0.41} \ M^{0.19} \ R^{1.97} \ \varepsilon^{0.58} \ \kappa_a^{0.78} \ (\text{rms err. 0.145}) \]

\( H \) is a factor used to express the degree of enhancement, that might be expected over the current mean prediction (i.e. it is usually 1), 

\( I_p \) is the plasma current (MA), \( B_T \) is the toroidal field on the plasma geometric axis (in T), \( P \) is the total net power crossing the separatrix from internal and external sources (MW), 

\( n_e \) is the geometric mean electron density \((10^{19} \text{ m}^{-3})\), \( M \) is the atomic mass of the plasma fuel (in AMU), \( R \) and \( a \) are plasma major and minor horizontal radii (m), 

and \( \kappa_a = S / (\pi a^2) \) with \( S \) being the plasma cross-sectional area.
Unmagnetized plasma

\[ D \equiv \chi_{||} = \frac{(\Delta x)^2}{2 \Delta t} \approx v \lambda_{\text{mfp}}^2 \approx \frac{v_{\text{th}}^2}{v} \]

Magnetized plasma

\[ D \equiv \chi_{\perp} = \frac{(\Delta x)^2}{2 \Delta t} \approx v \rho_L^2 \approx v_{\text{th}}^2 \frac{v}{\Omega_{L}^2} \]

\[ \Omega_L = \frac{e B}{m c} \]

\[ \chi_{\perp} = \frac{v T c^2}{e^2 B^2} \sim \frac{1}{B^2} \] magnitude and scaling often not correct!
Kinetic theory and transport truncations

Advantages:
- classical
- non-relativistic
- close to ideal

Disadvantages:
- geometry
- long mean free paths
- many scales
- linear
Kinetic and transport theories

Liouville equation → BBGKY hierarchy
→ truncation(s) in the kinetic regime
→ plasma kinetic equation(s)
  (Vlasov, Landau-Fokker-Planck, Balescu-Lenard)
→ averaging over gyromotion
  (drift-kinetic, gyro-kinetic equations)

\[
\frac{\partial f}{\partial t} + Df = J(f | f')
\]

Kinetic equation → hierarchy of moment equations
→ transport truncations
→ transport equations/coefficients

\[
\text{MHD} u_{E \times B} \sim v_{th}: f \approx f_M(\bar{v} - \bar{u}) + O(\delta), \delta = \rho / L
\]
\[
\text{drift} u_{E \times B} \sim \delta v_{th}: f \approx f_M(\bar{v})\left[1 + 2 \frac{\vec{u} \cdot \vec{v}}{v_{th}^2}\right] - \bar{\rho} \cdot \nabla f_M + O(\delta^2), \delta = \rho / L
\]
Linear transport theory

The classical theory for a homogeneous and stationary magnetic field does not apply as a local theory of transport in a tokamak. Global geometric characteristics of the confining elements have a strong influence on transport. Three regimes of collisionality are characteristic of the neoclassical transport theory:

- the banana regime (electronic diffusion increases starting from zero)
- the plateau regime (diffusion almost independent of collisionality)
- the Pfirsch-Schlüter regime (diffusion increases with collisionality)

\[ \lambda_D \] Debye length, \( \lambda_{\text{mfp}} \) mean free path, \( L_H \) hydrodynamic length, \( \rho_L \) Larmor radius

\[ \lambda_D \ll \rho_L \] : Landau type collision term, \( \rho_L \ll L_H \) : guiding center transport theory

\[ \lambda_D \ll \rho_L \ll \lambda_{\text{mfp}} \ll L_H \] : short mean free path regime (collisional regime)

\[ \lambda_D \ll \rho_L \ll L_H \ll \lambda_{\text{mfp}} \] : long mean free path regime (collisionless regime)
Neoclassical (linear) transport coefficients

Cross-coefficients lead to effects which have no classical or Pfirsch-Schlüter counterparts, e.g.

- a parallel electric field produces an inward radial electron flux
- a parallel electric field produces outward electron and ion heat fluxes
- a parallel electric current is produced by an radial ion heat flux, by a radial electron/ion temperature gradient
- radial pressure gradients may drive (bootstrap) currents

Nevertheless, open problems still exist:
\[ \chi_i \text{ up to a factor } 10 \text{ different} \]
\[ \chi_e \text{ up to a factor } 10^{-3} \text{ different} \]
\[ D_a \text{ up to a factor } 10^2 \text{ different} \]
Further demonstration of malfunction of linear transport theory

Magnetic reconnection times

\[
\frac{d\vec{B}}{dt} = \frac{\eta c^2}{4\pi} \frac{\partial^2 \vec{B}}{\partial y^2}, \quad \text{resistive time } \tau_r = \frac{4\pi a^2}{\eta c^2}
\]

<table>
<thead>
<tr>
<th>Classical resistive time</th>
<th>Observed energy release time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tokamaks</td>
<td>1 -10 s</td>
</tr>
<tr>
<td>Solar flares</td>
<td>$10^4$ µy</td>
</tr>
<tr>
<td>Magnetospheric substorms</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Problems arise on all scales:

from laboratory experiments (e.g. $L \sim 10^2 \rho$) [electron and ion losses]

up to galaktic discs ($d \sim 10^{27} \rho$, $\rho \sim 1$ AU) [cosmic radiation]
\[ \partial_t \mathbf{\Gamma} = -\nabla \cdot (\mathbf{u} \mathbf{\Gamma}) - \nabla p - \nabla \cdot \mathbf{P} + \frac{e}{mc} \mathbf{\Gamma} \times \mathbf{B} + \frac{e}{m} \mathbf{E} + \frac{1}{m} \mathbf{R} \]

\[ X := \langle X \rangle + \delta X \]

\[ \langle \Gamma_x \rangle = \frac{1}{m\Omega} \langle R_y \rangle + \frac{1}{\Omega} \langle (\hat{b} \times \nabla \cdot \mathbf{P}) \cdot \hat{x} \rangle + \Gamma_{x \text{anomalous}} + \Gamma_{x \text{time-dependent}} \]

\[ \Gamma_{x \text{anomalous}} = \frac{c}{B} \langle \delta n \delta E_y \rangle + \frac{1}{B} \left\{ \langle \delta \Gamma \parallel \delta B_x \rangle - \langle \delta \Gamma_x \delta B \parallel \rangle \right\} \]

Nonlinear gyrokinetics \(\rightarrow\) advanced computation
Advances in Computational Plasma Physics

GFlops/s (peak)

- Very idealized turbulence modelling
- 3D MHD turbulence, modellings over finite sections of a tokamak
- Numerical tokamak

Jahr

Analytical Approaches

Weak Turbulence Theory

\[ \Psi = L \Psi + \frac{1}{2} M \Psi \Psi, \quad L = i \Omega - \gamma, \text{ characteristic frequency } \Omega \]

\[ C_2(t,t') = \langle \delta \Psi(t) \delta \Psi(t') \rangle \] [correlation function]

Multiple scale analysis, \( \Omega_k + \Omega_p + \Omega_q \equiv \Delta \Omega, \quad \bar{k} + \bar{p} + \bar{q} = 0 \)

\[ \left( \frac{\partial}{\partial t_1} - 2\gamma \right) C_2 = M^2 \delta(\Delta \Omega) C_2^2 \]

Fluctuation spectrum: \( \langle \delta \varphi_k \delta \varphi_{k'}^* \rangle = n(k,t) \delta(k-k') \)

Wave kinetic equation: \( \frac{\partial n(k,t)}{\partial t} = I(k,t) + \Gamma(k) n(k,t) \)
The closure problem, renormalization, and structures

In reality, the resonance is broadened
- terms through all orders are required
- renormalization is required
→ strong turbulence theory

DIA (Direct Interaction Approximation) closure
EDQNM (Eddy Damped Quasi-Normal Markovian) closure
RMC (Realizable Markovian Closure)
...
Zonal flows
Blob generation in a turbulent tokamak edge
...

Intermittency

Coherent structures
Scaling Laws

Continuity equation in k-space \( \frac{\partial E(k)}{\partial t} + \frac{\partial P(k)}{\partial k} = 0 \)

\[
\begin{bmatrix}
E
\end{bmatrix} = \begin{bmatrix}
gL^2 \\
T^2
\end{bmatrix} \quad \begin{bmatrix}
E(k)
\end{bmatrix} = \begin{bmatrix}
\rho L^{6-d} \\
T^2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\rho
\end{bmatrix} = \begin{bmatrix}
g \\
L^3
\end{bmatrix} \quad \begin{bmatrix}
P(k)
\end{bmatrix} = \begin{bmatrix}
\rho L^{5-d} \\
T^3
\end{bmatrix}
\]

fluid turbulence: \( E(k) = \frac{\rho L^{6-d}}{T^2} \sim \rho^{1/3} P^{2/3} k^{d/3-8/3} \sim k^{-5/3} \) (Kolmogorov)

plasma turbulence: \( \omega = \omega(k) = \frac{1}{T} \rightarrow \text{non-dimensional variable } \xi = \frac{P k^{5-d}}{\rho \omega^3} \)

\( E(k) \sim \rho \omega^2(k) k^{d-6} f(\xi) \rightarrow \text{more freedom!} \)

e.g. \( f(\xi) \sim \xi^{1/3} \)
Heurist methods (small scale fluctuations)

\[ \Gamma_r \approx \langle \delta n \delta u_r \rangle \hat{=} -D \frac{\langle n \rangle}{L_n} \quad \text{[electrostatic turbulence]} \]

\[ \frac{\gamma_k}{\omega_k} \sim \delta_k, \quad \omega_k \approx \frac{ckT}{eBL_n}, \quad \delta n_k \approx \langle n \rangle \frac{e\phi_k (1-i\delta_k)}{T}, \quad \delta u_r \approx ick \frac{\phi_k}{B} \quad \text{[drift wave]} \]

\[ \Gamma_r \approx \langle n \rangle \frac{cT}{eB} \sum_k k\delta_k \left| \frac{\delta n_k}{\langle n \rangle} \right|^2 \]

\[ \nabla \delta n \hat{=} ik \delta n_k \sim \frac{\langle n \rangle}{L_n} \quad \text{[saturation]} \]

\[ D \approx \frac{\gamma_{\text{max}}}{k_{\text{max}}^2} \]
Mixing length argument (Prandtl)

\[ D \approx \frac{L_r^2}{\tau_c} \]

\( L_r \) mode radial correlation length, \( \tau_c \) autocorrelation time for turbulent fields

small scale turbulence: \( L_r \approx \rho_s \sim \rho_i \) [thermal ion Larmor radius]

\[ \tau_c^{-1} \approx \omega_T^* = \frac{k_o T}{B L_T} \sim \frac{v_{ti}}{a} \] [diamagnetic frequency], \( k_o \approx \rho_i^{-1} \)

\[ D \sim \rho_i^2 \omega_T^* \sim D_{\text{Bohm}} \frac{\rho_i^*}{L_T^*} \sim T^{3/2} B^{-2} \] [gyro-Bohm]

large scale turbulence: e.g. \( L_r \approx \sqrt{a \rho_s} \) through self-organization

\[ D \sim \rho_i v_{ti} \sim D_{\text{Bohm}} \left[ \frac{m^2}{s} \right] = \frac{T[\text{eV}]}{16 B[\text{T}]} \sim T B^{-1} \] [Bohm]

e.g. gyro-Bohm in H-Mode, Bohm or Goldstone regime \( D \sim \rho_i^{1/2} \) in L-Mode
Self-organization conjecture

Let a system be modeled by a nonlinear partial differential equation, with dissipation, for the field $u$. The system contains (at least) two quadratic or higher-order conserved quantities in the absence of dissipation. One of the conserved quantities, let us say $A(u)$, decays faster than the other(s), e.g. $B(u)$. The modal cascade in the quantity $B$ is towards small wave-numbers. Then, the field is expected to reach a quasi-stationary state which minimizes $A$ for constant $B$, i.e.

$$\delta A - \lambda \delta B = 0$$
Effective parallel transport in the presence of magnetic fluctuations

\[
\frac{\partial T}{\partial t} = \kappa_\parallel \left( \hat{n} \cdot \nabla \right)^2 T
\]

\[
\hat{n}_0 = \frac{\vec{B}_0}{B_0}, \quad \kappa_\parallel = \frac{v_{th}^2}{\nu} \gg \kappa_\perp
\]

\[
\vec{B} = \vec{B}_0 + \delta \vec{B}, \quad \vec{b} = \frac{\delta \vec{B}}{B}
\]

\[
\frac{\partial \langle T \rangle}{\partial t} = \kappa_\parallel \langle b^2 \rangle \frac{\partial^2 \langle T \rangle}{\partial x^2_\perp} + \kappa_\parallel \left( \frac{d}{dx_\perp} \left( \hat{n}_0 \cdot \nabla \right) \delta T \right)
\]

effective parallel transport
Status of analytical turbulence predictions

Self-consistent models
- are generally very difficult to solve
- are often good for scaling arguments

Heuristic models
- are often ad hoc
- are good for rough interpretations
- are, strictly speaking, not predictive

Non-locality, intermittency, interaction with coherent structures, … are open problems
„Manageable“ stochasticity

„Old“ paradigm: Stochasticity is
- mainly caused by plasma instabilities in high-dimensional phase space
- always counter-productive, i.e. enhanced losses
(micro-instabilities, MHD instabilities, ripple losses, …
due to thermodynamic forces and currents)

Nonlinear dynamics: Already a few degrees of freedom system may become stochastic
(Field-errors due to confinement, heating, shaping and correction coils,
MHD control coils (RWM) and boundary layer coils, Ergodic divertor coils, …)

„New“ paradigm: Stochastization may have positive aspects
- manageable in some respect
- ELM mitigation, particle exhaust, zonal flows, …

Tore-Supra, Island divertor in stellarators, DIII-D, Textor, JET (ASDEX-U, ITER): external sources for magnetic fluctuations
TORE SUPRA
DIII-D configuration
A unique feature of TEXTOR is the Dynamic Ergodic Divertor (DED). This device consists of a set of 16 helical coils mounted on the high-field side of the torus inside the vacuum vessel. The coils can be connected to produce perturbation fields with the fundamental mode numbers \( m/n = 12/4, 6/2, \) and \( 3/1. \) The coils can be supplied with dc current yielding a static perturbation field, or with ac currents resulting in a rotating field. The frequency of the rotating perturbation field can be either low (\( \sim 50 \text{ Hz} \)) or high frequency in the range 1 kHz to 10 kHz. These frequencies are of the order of the electron diamagnetic drift frequency for TEXTOR discharge conditions. The maximum coil current depends on the frequency and can be up to 15 kA.
Blanket coils: $2 \times 6 \times (5+5): 20\text{kA/11turns}$

9 mid-ports coils:
$150\text{kA/11turns}$

36 external coils:
$200-300\text{kA}$

Larger current, but better accessibility!
DIII-D results

Type-I ELMs are suppressed with resonant magnetic perturbations.
Toroidal spin-up of the plasma in TEXTOR (m/n=3/1)

\[ \omega_\phi = \frac{v_\phi}{R} \]

\[ E_r \leftrightarrow v_\phi \times B_\theta \]

Finken et al, PRL 2005
Runaway escape rates

slow (appr. 5 MeV)

fast (appr. 25-30 MeV)

\[
N(t) \approx N_0 \left( N_1 + \exp\{-\lambda t\} \right)
\]

\[
\lambda \sim -(E - E_c)
\]
Heat flux patterns (TEXTOR [top], DIII-D [bottom])

- **TEXTOR** (top): Shows a heat flux pattern with labeled inner and outer regions.
- **DIII-D** (bottom): Displays a heat flux pattern with a color scale indicating temperature levels from 0 to 220.
Semi-analytical treatments

\[ \vec{B} = \nabla \psi \times \nabla \theta - \nabla \Psi_p \times \nabla \varphi \]  
[Clebsch Representation]

\[ \vec{B} = \nabla \times \vec{A}, \quad \vec{A} = A_\rho \nabla \rho + A_\theta \nabla \theta + A_\zeta \nabla \zeta, \quad \frac{\partial G(\rho, \theta, \zeta)}{\partial \rho} = A_\rho \]

\[ \psi = A_\theta - \frac{\partial G(\rho, \theta, \zeta)}{\partial \theta}, \quad -\psi_p = A_\zeta - \frac{\partial G(\rho, \theta, \zeta)}{\partial \zeta} \]

\[ \Psi_p \equiv H = \int \frac{d\psi}{q(\psi)} \]

\[ \Psi_p \equiv H = \int \frac{d\psi}{q(\psi)} + H_1(\psi, \theta, \varphi) \]
Hamiltonian system

\[ \vec{B} = \nabla \psi \times \nabla \theta - \nabla H \times \nabla \varphi \]

\[ d\psi = \vec{B} \cdot \nabla \psi = - (\nabla H \times \nabla \varphi) \cdot \nabla \psi = - \left( \frac{\partial H}{\partial \theta} \nabla \theta \times \nabla \varphi \right) \cdot \nabla \psi = - \frac{\partial H}{\partial \theta} (\nabla \psi \times \nabla \theta) \cdot \nabla \varphi \]

\[ d\theta = \vec{B} \cdot \nabla \theta = - (\nabla H \times \nabla \varphi) \cdot \nabla \theta = - \left( \frac{\partial H}{\partial \psi} \nabla \psi \times \nabla \varphi \right) \cdot \nabla \theta = \frac{\partial H}{\partial \psi} (\nabla \psi \times \nabla \theta) \cdot \nabla \varphi \]

\[ d\varphi = \vec{B} \cdot \nabla \varphi = (\nabla \psi \times \nabla \theta) \cdot \nabla \varphi \]

Hamiltonian equations of motion

\[ \frac{d\psi}{d\varphi} = - \frac{\partial H}{\partial \theta} \quad \frac{d\theta}{d\varphi} = \frac{\partial H}{\partial \psi} \]
Resonances

\[ H(\vec{J}, \vec{\theta}) = H_0(\vec{J}) + \delta H(\vec{J}, \vec{\theta}) \]

\[ \frac{\partial H}{\partial \vec{J}} = \dot{\vec{\theta}} = \vec{\Omega}(\vec{J}) + \frac{\partial \delta H}{\partial \vec{J}} , \quad \frac{\partial H}{\partial \vec{\theta}} = -\vec{J} = \frac{\partial \delta H}{\partial \vec{\theta}} \]

\[ \delta H(\vec{J}, \vec{\theta}) = \sum \bar{m} e^{i\bar{m} \cdot \vec{\theta}} H_{\bar{m}} , \quad \vec{\theta} \approx \vec{\theta}_0 + \vec{\Omega}t \]

\[ \Delta \vec{J} = -\int dt \frac{\partial \delta H}{\partial \vec{\theta}} \approx -i \sum \bar{m} H_{\bar{m}} e^{i\bar{m} \cdot \vec{\theta}_0} \left( e^{i\bar{m} \cdot \vec{\Omega}t} - 1 \right) \frac{\bar{m} \cdot \vec{\Omega}}{\bar{m} \cdot \vec{\Omega}} \]

\[ \bar{m} \cdot \vec{\Omega}(\vec{J}) = 0 \quad [\text{resonances}] \]
Resonances, Chirikov overlap criterion, butterfly effect

Perturbation terms $\varepsilon H_{mn}(\psi) \cos(m\theta + n\phi + \chi_{m0})$ are resonant on the rational magnetic surfaces $\psi_{mn}$ with $q(\psi_{mn}) = m/n \rightarrow$ creating a chain of islands

\[
W_{mn} = 4 \left| \frac{\varepsilon H_{mn}(\psi_{mn})}{dq^{-1}/d\psi} \right|^{1/2} \quad \text{[island width]}
\]

\[
\sigma_{\text{Chir}} = \frac{W_{mn} + W_{m+1n}}{2|\psi_{m+1n} - \psi_{mn}|} \geq 1 \quad \text{[Chirikov parameter $\rightarrow$ onset of chaos]}
\]

TEXTOR DED:
The conventional method to determine the magnetic field topology is numerical field line tracing, but symplectic mappings are faster!

\[(\vartheta_{k+1}, \psi_{k+1}) = \hat{M}(\vartheta_k, \psi_k)\]

\[S_k = S(\vartheta, \Psi, \varphi; \varepsilon) \biggr|_{\varphi=\varphi_k}\]

\[
\begin{align*}
\Psi_k &= \psi_k - \varepsilon \frac{\partial S_k}{\partial \vartheta_k}, \\
\Theta_k &= \vartheta_k + \varepsilon \frac{\partial S_k}{\partial \Psi_k},
\end{align*}
\]

\[
\begin{align*}
\Psi_{k+1} &= \psi_{k+1}, \\
\Theta_{k+1} &= \Theta_k + \frac{(\varphi_{k+1} - \varphi_{k+1})}{q(\Psi_k)},
\end{align*}
\]

\[
\begin{align*}
\psi_{k+1} &= \Psi_{k+1} + \varepsilon \frac{\partial S_{k+1}}{\partial \vartheta_{k+1}}, \\
\vartheta_k &= \Theta_{k+1} - \varepsilon \frac{\partial S_{k+1}}{\partial \Psi_{k+1}}.
\end{align*}
\]
Poincaré plots

Comparison of correct symplectic (left) versus non-symplectic (right) map

Comparison of Poincaré plots: Field line tracing (left) versus symplectic map (right)
Characterization of chaotic systems

\[(\mathcal{G}_{k+1}, \psi_{k+1}) = \hat{M}(\mathcal{G}_k, \psi_k)\]

\[
d\tilde{I}_k = \left( \frac{d\psi_k}{d\mathcal{G}_k} \right) , \quad d\tilde{I}_{k+1} = J_k d\tilde{I}_k , \quad J_k = \begin{pmatrix} \frac{\partial \psi_{k+1}}{\partial \psi_k} & \frac{\partial \psi_{k+1}}{\partial \mathcal{G}_k} \\ \frac{\partial \mathcal{G}_{k+1}}{\partial \psi_k} & \frac{\partial \mathcal{G}_{k+1}}{\partial \mathcal{G}_k} \end{pmatrix}
\]

\[
\left| \begin{array}{cc} \frac{\partial \psi_{k+1}}{\partial \psi_k} - \lambda & \frac{\partial \psi_{k+1}}{\partial \mathcal{G}_k} \\ \frac{\partial \mathcal{G}_{k+1}}{\partial \psi_k} & \frac{\partial \mathcal{G}_{k+1}}{\partial \mathcal{G}_k} - \lambda \end{array} \right| = 0 , \quad \lambda^{(k)} = \max(\lambda_1^{(k)}, \lambda_2^{(k)})
\]

\[
\sigma = \lim_{N \to \infty} \frac{1}{N} \ln \prod_{k=1}^{N} \lambda^{(k)} > 0 \quad \text{[global Lyapunov exponent for unstable orbits]}
\]

\[
\sigma_N = \frac{1}{N} \ln \prod_{k=1}^{N} \lambda^{(k)} > 0 \quad \text{[finite-time Lyapunov exponent for finite connection lengths]}
\]

\[
L_N = \frac{l}{\sigma_N} \quad \text{[local e-folding length, } l \text{ length of field line per one map step]}
\]
Characterization of chaotic systems

\[(\vartheta_{k+1}, \psi_{k+1}) = \hat{M}(\vartheta_k, \psi_k)\]

Kolmogorov length

\[L_K(\rho, N) = \overline{L}_N = \frac{l}{\overline{\sigma}_N}\] [averaged over magnetic surface of radius \(\rho\)]

\[\approx \pi q R_0 \left( \frac{\pi \sigma_{Chir}}{2} \right)^{-4/3}\] [quasi-classical approximation]

\[\langle |\Delta \vec{x}_\perp(\zeta)|^2 \rangle \sim |\Delta \vec{x}_\perp(0)|^2 \exp\left\{\frac{2\zeta}{L_K}\right\}\]
(Open) chaotic scattering system

Connection lengths → Laminar plots
CIII emission spectrum and field line structure

Laminar region: Connection length < Kolmogorov length
Quasi-linear field line diffusion

\[
\frac{d\psi}{d\zeta} = -\frac{\partial \Psi_p}{\partial \theta}, \quad \frac{d\theta}{d\zeta} = \frac{\partial \Psi_p}{\partial \psi}
\]

\[
H \equiv \Psi_p = H_0(\psi) + \varepsilon H_1(\theta, \psi, \zeta), \quad V_1 = -\frac{\partial H_1}{\partial \theta}
\]

\[
\frac{df}{dt} = \frac{\partial f}{\partial \zeta} + \dot{\theta} \frac{\partial f}{\partial \theta} + \dot{\psi} \frac{\partial f}{\partial \psi} = \frac{\partial f}{\partial \zeta} + \frac{\partial H}{\partial \psi} \frac{\partial f}{\partial \theta} - \frac{\partial H}{\partial \theta} \frac{\partial f}{\partial \psi}
\]

\[
\frac{\partial f_0}{\partial \zeta} = \frac{\partial}{\partial \psi} \left( D \frac{\partial f_0}{\partial \psi} \right)
\]

\[
D \equiv D(\psi) = \left\langle \int_0^{\zeta_0} d\zeta_0' V_1\left[ \theta - \Omega \zeta_0', \psi, \zeta_0 - \zeta_0' \right] V_1\left[ \theta, \psi, \zeta_0 \right] \right\rangle_{\theta, \zeta_0}
\]
Magnetic diffusion coefficient

\[
D_{m(\text{magnetic})} = \int_0^\infty d\zeta \left\langle b_x [\tilde{x}_\perp (\zeta), \zeta] b_x [\tilde{x}_\perp (0), 0] \right\rangle \sim b^2 L_{\text{corr}}
\]

\[\Delta x < \lambda_\perp : \quad D_m \lambda_\parallel < \lambda_\perp^2 \rightarrow L_{\text{corr}} \approx \lambda_\parallel \rightarrow D_m \approx b^2 \lambda_\parallel\]

\[\Delta x > \lambda_\perp : \quad D_m \lambda_\parallel > \lambda_\perp^2 \rightarrow D_m L_{\text{corr}} \approx \lambda_\perp^2 \rightarrow D_m \approx \frac{b^2 \lambda_\perp^2}{D_m} \rightarrow D_m \approx b \lambda_\perp\]

\[
D_m \approx \begin{cases} 
  b^2 \lambda_\parallel & \text{for } b < \frac{\lambda_\perp}{\lambda_\parallel} \\
  b \lambda_\perp & \text{for } b > \frac{\lambda_\perp}{\lambda_\parallel}
\end{cases}
\]

Results without trapping:

\[D_m \approx b^2 \lambda_\parallel \quad (\text{for } \lambda_\perp \rightarrow \infty) \quad \text{[quasilinear]}\]

\[D_m \approx b \lambda_\perp \quad (\text{for } \lambda_\parallel \rightarrow \infty, \lambda_\perp \text{ finite}) \quad \text{[Kadomtsev-Pogutse]} \text{ trapping?}\]
Non-diffusive transport in the presence of hyperbolic fixed points

Stable and unstable manifolds of periodic hyperbolic fixed points
Heat deposition pattern

Laminar plots versus manifold strike points (see also the MASTOC criterion)

Experiment:

Theory:

$q -$ profile at the edge: \( q_a = \frac{2\pi B_0 \alpha^2}{\mu_0 R_0 I_p} \)

Manifolds select outer strike zones
Ab initio theory: A-Langevin equation model

\[ \dot{u}(t) = \frac{Ze}{mc} \mathbf{u}(t) \times \mathbf{B}(t) - \nu \mathbf{u}(t) + a(t) \]

\[ \mathbf{B} = B_0 (b_0 e_z + b) \]

Lagrangian correlator + Green-Kubo formalism: ODE for diffusion coefficient
V-Langevin approach (Balescu and co-workers)

Guiding center approach

\[
\begin{align*}
\frac{dx_p(t)}{dt} &= b_x [x_p(t), y_p(t), z_p(t)] \frac{dz_p(t)}{dt} + \eta_{\perp x}(t) \\
\frac{dy_p(t)}{dt} &= b_y [x_p(t), y_p(t), z_p(t)] \frac{dz_p(t)}{dt} + \eta_{\perp y}(t) \\
\frac{dz_p(t)}{dt} &= \eta_{\parallel}(t)
\end{align*}
\]
Trapping and finite Larmor radii

Comparison: island width \( w_{m,n} \)

\[
w_{m,n} = \Lambda(\theta) \left| \frac{r_{m,n} b_{r(m,n)} q_{m,n}}{mb_\theta \partial q_{m,n}/dr} \right|^{1/2} \sim \sqrt{b_{r(m,n)}}
\]

and Larmor radius \( \rho \)

Radial position of one trajectory

\[
\dot{u}(t) = \frac{Ze}{mc} u(t) \times B(t) - \nu u(t) + a(t)
\]

\[
B = B_0 (b_0 e_z + b)
\]
Statistical physics of field line diffusion

Taylor-Green-Kubo formula: \[ D = \frac{\langle \ddot{x}^2 \rangle}{2td} = \frac{1}{d} \int_0^\infty d\tau \, \langle \vec{v}[\vec{x}(\tau), \tau] \cdot \vec{v}[0,0] \rangle \]

Lagrange correlation

\[ D_{m(\text{magnetic})} = \int_0^\infty d\zeta \, \langle b_x[\vec{x}_\perp(\zeta), \zeta] \rangle \langle b_x[\vec{x}_\perp(0), 0] \rangle \sim b^2 L_{\text{corr}} \]

\[ L_{rs}(\zeta) = \langle b_r[\vec{x}_\perp(\zeta), \zeta] \rangle \langle b_s[\vec{x}_\perp(0), 0] \rangle \]

\[ B_{mn} = \langle b_m[\vec{x} + \vec{r}] \rangle \langle b_n[\vec{x}] \rangle \approx \beta^2 \left[ \left(1 - \frac{r^2}{\lambda^2} \right) \delta_{mn} + \frac{r_m r_n}{\lambda^2} \right] \exp \left\{ - \frac{z^2}{2\lambda||^2} - \frac{r^2}{2\lambda^2} \right\} \]

Corrsin: \[ L_{rs}(\zeta) \approx \int d^2 r_\perp B_{rs}(\vec{r}_\perp, \zeta) \langle \delta(\vec{r}_\perp - \vec{x}_\perp(\zeta)) \rangle \]

\[ L(\zeta) \approx \beta^2 e^{-\zeta^2/2\lambda||^2} \frac{\lambda_\perp^4}{\left[ \lambda_\perp^2 + 2\int_0^\zeta d\zeta' (\zeta - \zeta') L(\zeta') \right]^2} \]

\[ D_m \approx \begin{cases} \frac{b^2}{2} \lambda_\parallel & \text{for } \lambda_\parallel \to \infty \\ \frac{b}{2} \lambda_\perp & \text{for } \lambda_\parallel \to \infty \end{cases} \]
Quasilinear particle diffusion properties

\[ \sigma^2 = 2D_{\text{particle}} t^\gamma ; \quad \gamma = 1 \text{ diffusive} \]

⊥ non-diffusive (\(\parallel\) diffusive, no deviation from magnetic field line)

\[ \sigma^2 = 2D_{\text{magnetic}} z , \quad \sigma^2 = 2D_{\text{magnetic}} \sqrt{2\chi t} \rightarrow \frac{\sigma^2}{t} \sim \frac{1}{\sqrt{t}} \xrightarrow{t \to \infty} 0 \]

Quasilinear particle diffusion (Jokipii & Parker)

\[ D_{\perp \text{particle}} \approx \frac{\langle (\Delta x)^2 \rangle}{\Delta t} \approx \frac{D_{\text{magnetic}} \lambda_{\parallel}}{\lambda_{\parallel} / v_{\text{thermal}}} = D_{\text{magnetic}} v_{\text{thermal}} \approx \beta^2 \lambda_{\parallel} v_{\text{thermal}} \quad [\text{collisionless}] \]

\[ D_{\perp \text{particle}} \approx \frac{\langle (\Delta x)^2 \rangle}{\Delta t} \approx \begin{cases} D_{\text{magnetic}} \frac{L_{\text{corr}}}{\tau_c} \approx \beta^2 \lambda_{\parallel}^2 v_{\text{collision}} \approx \beta^2 D_{\parallel \text{particle}} & [\text{collisional fluid limit, } \lambda_{\parallel} \approx L_{\text{corr}} \approx v_{\text{thermal}}/v_{\text{collision}} ] \\ \cdots \end{cases} \]

see more during ab initio calculations!
Rechester-Rosenbluth („oversimplified“)

\[ \left< (\Delta x)^2 \right> \approx \beta^2 \lambda_{||} L_K \]

decorrelation time: \[ t_i \approx \frac{L_K^2}{\chi_{||}} \]

\[ D_{\text{particle}} \approx \frac{\left< (\Delta x)^2 \right>}{t_i} \]

\[ D_{\text{particle}} \approx \frac{\beta^2 \lambda_{||} \chi_{||}}{L_K} \]
Rechester-Rosenbluth coefficient (weakly collisional)

\[ \lambda_{mfp} \ll L_K < L_{\text{diffusion}} \]

\[ \left\langle (\Delta x)^2 \right\rangle_{L_K} \approx \chi_\perp T_{L_K} = \frac{L_K^2}{\chi_\parallel} \quad \text{with} \quad \chi_\perp = \frac{v^2}{\Omega^2} \chi_\parallel, \quad \chi_\parallel = \frac{v_{th}^2}{2v}, \quad v T_{L_K} > 1 \]

magnetic field separation: \[ \lambda_\perp^2 \approx \left\langle (\Delta x)^2 \right\rangle_{L_K} \exp \left\{ \frac{2L_{\text{diffusion}}}{L_K} \right\} \rightarrow L_{\text{diffusion}} \approx L_K \ln \left( \frac{\lambda_\perp}{\sqrt{\left\langle (\Delta x)^2 \right\rangle_{L_K}}} \right) \]

\[ \left\langle (\Delta x)^2 \right\rangle_{L_{\text{diffusion}}} \approx D_m L_{\text{diffusion}}, \quad D_{\text{particle}} \approx \frac{\left\langle (\Delta x)^2 \right\rangle_{L_{\text{diffusion}}}}{L_{\text{diffusion}}^2} \frac{L_{\text{diffusion}}}{L_K} \]

\[ D_{\text{particle}}^{RR} \approx \frac{D_m \chi_\parallel}{L_K \ln \left( \frac{\lambda_\perp}{L_K \sqrt{\chi_\parallel}} \right)} \sim \frac{D_m \chi_\parallel}{\chi_\parallel} \beta^2 K^2 \quad \text{with} \quad K \approx \frac{\beta \lambda_\parallel}{\lambda_\perp} \quad \text{and} \quad L_K \approx \sqrt{\frac{2}{\pi}} \frac{\lambda_\perp^2}{4 \beta^2 \lambda_\parallel} \]
Kadomtsev-Pogutse coefficient (strongly collisional)

\[ \langle (\Delta x)^2 \rangle_l \approx \beta^2 \lambda_\parallel l \quad \text{with} \quad l^2 \approx \chi_\parallel t_l \]

decorrelation time: \( t_l \approx \frac{\lambda_\perp^2}{\chi_\perp} \)

\[ D_{\text{particle}} \approx \frac{\langle (\Delta x)^2 \rangle_l}{t_l} \]

\[ D_{\text{particle}} \approx \frac{\beta^2 \lambda_\parallel}{\lambda_\perp} \sqrt{\chi_\parallel \chi_\perp} \quad \sim \beta K D_{\text{Bohm}} \]

with \( K \approx \frac{\beta \lambda_\parallel}{\lambda_\perp} \) and \( D_{\text{Bohm}} \approx \frac{1}{16} \sqrt{\chi_\parallel \chi_\perp} \sim \frac{cT}{eB} \)
Comparison (simplified)

Rechester-Rosenbluth:

\[
\left\langle (\Delta x)^2 \right\rangle_{L_K} \approx D_m L_K = D_{\text{particle}} T_K
\]

\[
T_K \approx \frac{L_K^2}{\chi_{||}}
\]

\[
D_{\text{particle}}^{RR} \approx \frac{\beta^2 \lambda_{||} \chi_{||}}{L_K} \sim \frac{\beta^4 \lambda_{||}^2 \chi_{||}}{\lambda_{\perp}^2}
\]

\[
T_K \ll t_l: \quad \nu < \Omega \beta K
\]

Kadomtsev-Pogutse:

\[
\left\langle (\Delta x)^2 \right\rangle_{l_{\text{coll}}} \approx D_m l_{\text{coll}} = D_{\text{particle}} t_l
\]

\[
t_l \approx \frac{l_{\text{coll}}^2}{\chi_{||}}, \quad t_l \approx \frac{\lambda_{\perp}^2}{\chi_{\perp}} \rightarrow l_{\text{coll}} \approx \sqrt{\frac{\chi_{||}}{\chi_{\perp}}} \lambda_{\perp}
\]

\[
D_{\text{particle}}^{KP} \approx \frac{\left\langle (\Delta x)^2 \right\rangle}{t_l} \approx \frac{\beta^2 \lambda_{||}}{\lambda_{\perp}} \sqrt{\frac{\chi_{||}}{\chi_{\perp}}}
\]

\[
T_K \gg t_l: \quad \nu > \Omega \beta K
\]
Running diffusion coefficient for $K<1$ (Langevin theory)

Parameters: $\nu$, $\lambda_{\parallel}$, $\lambda_{\perp}$, $\beta$, $\Omega$

$$D \sim \beta^2 \lambda_{\parallel} v_{th} \sim K \; (\text{Jokipii&Parker}) \text{ or } D \sim \beta^2 \chi_{\parallel} \; (\text{fluid}) \text{ or } \chi_{\parallel} \; (\beta \to 0, \nu \neq 0)$$
Finite Larmor radius effects for $K<1$

Finite Larmor radii decorrelate the particles from exponentially diverging magnetic field lines.

Monte Carlo simulations support the analytical predictions.
The role of large Kubo numbers

\[ b(x, z) = \nabla \phi(x, z) \times e_z \]

Kubo number \( K \):

\[ K = \frac{V \tau_c}{\lambda_c} \]

\( K \approx \frac{\beta \lambda_{||}}{\lambda_{\perp}} \leq 1: \) Corrsin

\( K \gg 1: \) DCT
Trapping
Percolation
Percolation limit $K >> 1$

\[ D \sim \frac{1}{K^{0.7}} \]
D → 0 in the percolation limit K >> 1

naive (wrong!) argument: \( \lambda_\parallel \to \infty : D_{magnetic} \sim \beta \lambda_\perp \sim K^0 \)

\[
\frac{dx}{dz} = \frac{b_x^{\text{trapping}} + b_x^{\text{random}}}{B_0}, \quad \frac{dy}{dz} = \frac{b_y^{\text{trapping}} + b_y^{\text{random}}}{B_0}, \quad \vec{b}^{\text{trapping}} = \nabla \times \hat{z}, \quad a \approx A \exp \left\{ -\frac{r^2}{2\sigma^2} \right\}
\]

\[
\frac{d^2x}{dz^2} \approx -\kappa^2 x, \quad \kappa = \frac{a}{B_0 \sigma^2}
\]

\[
\langle \Delta r^2 \rangle = \frac{1}{B_0^2} \int \int dz' dz'' \langle b_r^{\text{random}}(z') b_r^{\text{random}}(z'') \rangle \sim \frac{2z}{B_0^2} P_{xx}(\kappa)
\]

with \( P_{xx}(k) = \frac{1}{2\pi} \int B_{xx}(\Delta z) \exp\{-ik\Delta z\} d\Delta z \) since \( b_r^{\text{random}}(z) \sim b_x^{\text{random}}(z) \cos(\kappa z) \)

example: Kolmogorov spectrum \( P_{xx}(k) \approx \frac{C}{\left[1 + (k\lambda_\parallel)^2\right]^{5/6}} \)

\[
D_{rr} \approx D_{\text{quasilinear}} \frac{P_{xx}(\kappa)}{P_{xx}(0)} \sim \left\{ \begin{array}{ll}
D_{\text{quasilinear}} & \text{for } \kappa \lambda_\parallel \ll 1 \\
\kappa^{-5/3} \to 0 & \text{for } \kappa \lambda_\parallel \gg 1
\end{array} \right.
\]

\[
\kappa \lambda_\parallel = \frac{a \lambda_\parallel}{B_0 \sigma^2} \sim \frac{a}{B_0 \sigma} \frac{\lambda_\parallel}{\sigma} \sim \beta \frac{\lambda_\parallel}{\lambda_\perp} \sim K \text{ [Kubo number]}
\]
DCT (decorrelation trajectory method)

Subensemble $\phi(0,0) = \phi^0, \quad \vec{v}(0,0) = \vec{v}^0$

Lagrange correlator $L_{ij} = \int d\phi^0 d\vec{v}^0 P(\phi^0, \vec{v}^0) v_i^0 \langle \nu_j(\vec{r}(t),t) \rangle^S$

Eulerian mean velocity via conditional probability:

$\langle \nu_j(\vec{r},t) \rangle^S = \left[ \phi^0 E_{\phi j}(\vec{r}) + v_x^0 E_{xj}(\vec{r}) + v_y^0 E_{yj}(\vec{r}) \right] \exp\{-\theta / K\}$

$\equiv f_j(\vec{r}; \phi^0, \vec{v}^0) \exp\{-\theta / K\}, \quad \theta \sim t$

Space–time decorrelation trajectory:

$\frac{d}{dt} \bar{X}^S(t) = \bar{f}(\bar{X}^S; \phi^0, \vec{v}^0) \exp\{-\theta / K\} \equiv \bar{V}^S(\bar{X}^S(t),t), \quad \bar{X}^S(t = 0) = \bar{0}$

$L_{ij} = \int d\phi^0 d\vec{v}^0 P(\phi^0, \vec{v}^0) v_i^0 V_j^S(\bar{X}^S(t),t)$
Finite Larmor radius effects

Kadomtsev-Pogutse result $\beta \lambda_{\perp}$

Increase due to detrapping

Gyrocenter result

Finite Larmor radius corrections

Reduction due to decorrelation
Large Kubo numbers $K \gg 1$

Finite Larmor radii reduce the trapping effects (collisionless case)

Collisions reduce the trapping effects (collisional cases)
Percolative regime: Comparison with simulations

\[ \langle \Delta x^2(t) \rangle \]

- DCT approximation with Larmor radius correction
- Guiding center limit with DCT
Parallel (pitch angle) diffusion

**Effective collision frequency:**

\[ \nu \rightarrow \nu + \nu_{\text{fluctuation}}, \quad \beta = \frac{\delta B}{b_0}, \quad \Omega = \frac{eb_0}{mc} \]

\[ \nu_{\text{fluctuation}} \approx \beta^2 \frac{\lambda_\parallel \Omega^2}{\nu_{\text{thermal}}} \]

\[ \hat{=} \text{quasilinear} \]

**Quasilinear collisionless case**

\[ K \ll 1 \]
Parallel (pitch angle) diffusion

$K >> 1$

DCT versus Corrsin
SUMMARY

Closures (beyond Corrsin) in the presence of structures (incomplete chaos) seem to work well.

We have a quite complete picture of the parameter dependencies of stochastic transport coefficients.

The qualitative as well as quantitative effects of stochastic transport should be further evaluated in dynamical models for ergodic divertors.

Manageable stochasticity
• has many applications
• is a test basis for nonlinear plasma transport
• is ITER relevant

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Correct citations will appear in the written text!