Energetic particle modes: from bump on tail to tokamak

M. K. Lilley

Dept. of Earth and Space Sciences, Chalmers University of Technology, 41296 Göteborg, Sweden
Outline

• Single particle perspective
• Coherent motion → plasma modes
• Effect of fast particles
• Case study: Bump-on-tail
• Generalisation to 3D world
• Outstanding problems
Single particle perspective
Confinement: a first glance

- Static fields $\rightarrow$ constant particle energy ($E$)
- Weak spatial non-uniformity of field $\rightarrow$ ”constant” magnetic moment ($\mu$)
- Axisymmetry $\rightarrow$ constant toroidal angular momentum ($p_\phi$)

$$p_\phi = mR v_\phi + e\psi(r)$$

- Particles have finite excursion from flux surface due to drifts $\rightarrow$ bounded orbits
Confinement: more detailed

- Axisymmetry is an idealisation, e.g. Ripple effects
- Broken symmetry can lead to loss of confinement
- In general the EM fields are not static...there are many charged particles moving around
- Microscopic time varying fields break invariants of motion and lead to loss of confinement

\[ e\phi \ll k_B T \]

- These microscopic fields are the collisions which lead to diffusion of particles out of the tokamak
Coherent motion
Coherent plasma motion

- Coherent motion leading to waves only occurs if the plasma current responds in the same way as the fields, e.g. $E$ & $j \sim \sin(\omega t)$

- This only happens if the distribution of particles can be considered as stationary

- Not true in reality, but statistical description, i.e. continuous distribution function, allows this

- Good only when the plasma is sufficiently dense, need many particles per wavelength
Coherent plasma motion

- Waves need energy
- Tokamak not in thermodynamic equilibrium

\[ \nabla P = j \times B \]

- Current and density gradient drive waves, e.g. Kink, ballooning modes, typically low frequency
- Another source of free energy in fast particles – The waves are characterised by bulk plasma but are excited by the low density fast population
Fast particle driven modes – Soft nonlinearity – TAEs via ICRH on JET

#49447, probe H302: mode amplitude $\log(\|B(T)\|)$

-4
-5
-6
-7
-8
-9
-10
-11

frequency (kHz)

130 135 140 145 150 155 160

44.3 44.4 44.5 44.6

time (sec)
Fast particle driven modes – Rapid sweeping

The ms timescale of these events is much shorter than the energy confinement time in the plasma.

MAST data

JET data

NSTX data
Fast particle driven modes – Mixed – Beam driven CAEs on MAST
Fast particle driven modes – Particle loss in TFTR

Saturation of the neutron signal reflects anomalous losses of the injected beams. The losses result from Alfvénic activity.

K. L. Wong et.al PRL 66, 1874 (1991)
Coherent plasma motion

- Coherent motion of plasma can have a much larger effect than collisions
- Effect of waves on confinement of particles cannot be universally predicted
- Each case must be dealt with separately
- We will focus on instabilities driven by fast particles in this lecture
Effect of fast particles on waves
The Questions

- How does a low density population produce a large effect
- How do the fast particles produce such rich non linear evolution at different timescales
- How is it that the same modes driven by different particles look so different

ICRH drive (TAE-JET)

Heeter et.al PRL 85, 3177 (2000)

NBI drive (TAE-MAST)

Pinches et.al PPCF, 46, S47 (2004)
Resonance

- Special group of particles that strongly interact with a wave

\[ \text{force} \sim e^{i(k \cdot r - \omega t)} \rightarrow e^{i(k(x_0 + v_0 t) - \omega t)} \]

- \( v_0 = \omega / k \) gives non oscillating force on particle

- Provides a channel for energy to go from the source into the coherent motion of background not thermal motion

- Allows low density fast particles to pump/drive the wave
Marginal stability

- System evolves through a threshold

\[ |\gamma_L - \gamma_d| \ll \gamma_L, \gamma_d \]

- Collision times are comparable to growth times
Case study: Bump-on-tail
Bump on tail - Basic ingredients

- Particle injection and effective collisions, $v_{\text{eff}}$, create an inverted distribution of energetic particles $F_0(v)$
- Discrete spectrum of unstable electrostatic modes
- Instability drive, $\gamma_L \sim \frac{dF_0}{dv}$, due to wave-particle resonance ($\omega - kv = 0$)
- Background dissipation rate, $\gamma_d$, determines the critical gradient for the instability

$$\gamma_L = \gamma_d$$

$$v = \frac{\omega}{k}$$

$$m(v - \omega/k)$$

$$\omega_B$$
Bump on tail - Basic ingredients

- In wave frame the electric potential creates trapped and passing particles
  \[ \mathcal{E} = \frac{1}{2} m v^2 - |e| \phi \rightarrow v_{\text{sep}} = \sqrt{\frac{2}{m}} \sqrt{\phi_{\text{max}} - \phi} \]
- Separatrix is the trapped/passing boundary
- Motion in phase space is that of a pendulum with frequency \( \omega_B \) determined by amplitude of field \( \omega_B \sim \hat{E}^{1/2} \)

Diagram:
- Critical slope \( \gamma_L = \gamma_d \)
- \( v = \omega/k \) vs. \( V \)
- \( m(v - \omega/k) \)
- Separatrix

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No dissipation or collisions – Schematic
No dissipation or collisions – Schematic
No dissipation or collisions – Schematic

\[ v = \omega / k \]
No dissipation or collisions – Schematic

\[ \nu = \omega/k \]
No dissipation or collisions – Schematic

\[ v = \frac{\omega}{k} \]
No dissipation or collisions – Schematic

\[ v = \frac{\omega}{k} \]
No dissipation or collisions – Schematic

\[ v = \omega / k \]
No dissipation or collisions – Saturation level

\[ \frac{1}{2} m (\Delta v)^2 = -|e| \phi \rightarrow \Delta v \sim \sqrt{\hat{E}} \]

\[ m \frac{\omega}{k} \Delta v \times \frac{1}{2} \Delta v \frac{\partial F_0}{\partial \Delta v} \Delta v \]

energy release per particle

no. particles

\[ \hat{E}^{3/2} \]

fast particles energy

wave energy

- Wave grows until fast particle energy release cannot support the wave energy

\[ \rightarrow \omega_B \equiv \sqrt{\frac{|e| k\hat{E}}{m}} \sim \gamma_L \]
No dissipation or collisions – Saturation level

\[ \omega_B^2 \sim \hat{E} \]
No dissipation or collisions – Phase space plateaux

- Distribution is only strongly perturbed inside separatrix (black line)
Bump on tail – Key bits of physics

- Wave creates perturbations in velocity space around resonance.
- Mixing area is bounded by the separatrix, determined by bounce frequency. This separates trapped and passing particles.
- Bounce frequency is on the order of $\gamma_L$.
- $\gamma_L$ is small compared to wave frequency, which means the electric field is "small" so that perturbation of background is small which means linear.
Bump on tail - formalism

- Linear cold background with sinusoidal field
  \[ E = \frac{1}{2} \left[ \hat{E}(t) e^{i\zeta} + \text{c.c.} \right] \]
  \[ \zeta = kx - \omega t \]
  \[ u = kv - \omega \]
  \[ \frac{\partial V}{\partial t} = -\frac{e |E|}{m} - v_c V \]

- Kinetic fast particle population
  \[ F = F_0 + f_0 + \sum_{n=1}^{\infty} \left[ f_n \exp(i n\zeta) + \text{c.c.} \right] \]

- Current from cold background obtained perturbatively using smallness of wave growth and dissipation
  \[ \frac{\partial E}{\partial t} = -\frac{\omega}{\varepsilon_0 k^2} e \int f_1 du - \gamma_d \hat{E} \]
  \[ \gamma_d = v_c / 2 \]
Collisionality – First glance

• Marginal stability allows collisions to compete with mode growth

\[ |\gamma_L - \gamma_d| \sim v_{\text{eff}} \]

• Krook and diffusion have been studied

\[
\left. \frac{dF}{dt} \right|_{\text{coll}} = \beta (F - F_0) \quad \left. \frac{dF}{dt} \right|_{\text{coll}} = \frac{v^3}{k^2} \left( \frac{\partial^2 F}{\partial v^2} - \frac{\partial^2 F_0}{\partial v^2} \right)
\]

• Note: Krook is normally to mock up diffusion, but can actually be physical if collisions move particle immediately out of resonance (not typical in fusion conditions)

\[ v_{\text{eff}} = \max \{ \beta, v \} \]
Near threshold ordering

• Perturbative approach applied $\rightarrow$ time scales shorter than non-linear bounce period of the wave $\omega_B^{-1}$

$$\omega_B^2 = \frac{ek\hat{E}}{m}$$

• Can be maintained indefinitely if collision frequency is much larger than bounce frequency

• The distribution function will not be significantly perturbed:

$$F_0 \gg f_1 \gg f_0, f_2$$
Near threshold ordering

\[
\frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1}{\partial u} + \text{c.c} \right)
\]

\[
\frac{\partial f_1}{\partial t} + iuf_1 - \nu^3 \frac{\partial^2 f_1}{\partial u^2} + \beta f_1 = -\frac{ek}{2m} \hat{E} \frac{\partial}{\partial u} \left( F_0 + f_0 + f_2 \right)
\]

\[
\frac{\partial f_2}{\partial t} + 2iuf_2 - \nu^3 \frac{\partial^2 f_2}{\partial u^2} + \beta f_2 = -\frac{ek}{2m} \hat{E}^* \frac{\partial f_1}{\partial u} + O(\hat{E} f_3)
\]

\[f_1 \sim\]
Near threshold ordering

\[ \frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + \text{c.c} \right) \]

\[ \frac{\partial f_1}{\partial t} + iuf_1 - \nu^3 \frac{\partial^2 f_1}{\partial u^2} + \beta f_1 = -\frac{ek}{2m} \hat{E} \frac{\partial}{\partial u} \left( F_0 + f_0 + f_2 \right) \]

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\[ f_1 \sim \]
Near threshold ordering

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\frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + \text{c.c.} \right)
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Near threshold ordering

\[ \frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + \text{c.c.} \right) \]

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\[ f_1 \sim \gamma L \hat{E} \]
Near threshold ordering

\[ \frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + \text{c.c.} \right) \]

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\[ f_1 \sim \gamma_L \hat{E} \]
Near threshold ordering

\[
\begin{align*}
\frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 &= -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + c.c \right) \\
\frac{\partial f_1}{\partial t} + i uf_1 - \nu^3 \frac{\partial^2 f_1}{\partial u^2} + \beta f_1 &= -\frac{ek}{2m} \hat{E} \frac{\partial}{\partial u} \left( F_0 + f_0 + f_2 \right) \\
\frac{\partial f_2}{\partial t} + 2i uf_2 - \nu^3 \frac{\partial^2 f_2}{\partial u^2} + \beta f_2 &= -\frac{ek}{2m} \hat{E}^* \frac{\partial f_1}{\partial u} + O(\hat{E} f_3) \\
f_1 &\sim \gamma_L \hat{E}
\end{align*}
\]
Near threshold ordering

\[
\frac{\partial f_0}{\partial t} - v^2 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + \text{c.c.} \right)
\]

\[
\frac{\partial f_1}{\partial t} + iuf_1 - v^2 \frac{\partial^2 f_1}{\partial u^2} + \beta f_1 = -\frac{ek}{2m} \hat{E} \frac{\partial}{\partial u} (F_0 + f_0 + f_2)
\]

\[
\frac{\partial f_2}{\partial t} + 2iuf_2 - v^2 \frac{\partial^2 f_2}{\partial u^2} + \beta f_2 = -\frac{ek}{2m} \hat{E}^* \frac{\partial f_1}{\partial u} + O(\hat{E}f_3)
\]

\[f_1 \sim \gamma_L \hat{E}\]
Near threshold ordering

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\frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{\mathcal{E}} \frac{\partial f^*_1}{\partial u} + \text{c.c} \right)
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\[
\frac{\partial f_1}{\partial t} + iuf_1 - \nu^3 \frac{\partial^2 f_1}{\partial u^2} + \beta f_1 = -\frac{ek}{2m} \hat{\mathcal{E}} \frac{\partial}{\partial u} \left( F_0 + f_0 + f_2 \right)
\]

\[
\frac{\partial f_2}{\partial t} + 2iuf_2 - \nu^3 \frac{\partial^2 f_2}{\partial u^2} + \beta f_2 = -\frac{ek}{2m} \hat{\mathcal{E}}^* \frac{\partial f_1}{\partial u} + O \left( \hat{\mathcal{E}}^f_3 \right)
\]

\[
f_1 \sim \gamma_L \hat{\mathcal{E}} + \gamma_L c_3 \hat{\mathcal{E}}^3 + \ldots
\]
Near threshold ordering

\[ \frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + \text{c.c.} \right) \]

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\[ f_1 \sim \gamma_L \hat{E} + \gamma_L c_3 \hat{E}^3 + \ldots \]

\[ \frac{\partial \hat{E}}{\partial t} = -4 \frac{\omega}{k^2} \pi e \int f_1 du - \gamma_d \hat{E} \]
Near threshold ordering

\[ \frac{\partial f_0}{\partial t} - \nu^3 \frac{\partial^2 f_0}{\partial u^2} + \beta f_0 = -\frac{ek}{2m} \left( \hat{E} \frac{\partial f_1^*}{\partial u} + c.c \right) \]

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\[ f_1 \sim \gamma_L \hat{E} + \gamma_L c_3 \hat{E}^3 + \ldots \]

\[ \frac{\partial \hat{E}}{\partial t} \sim \gamma_L \hat{E} \left( 1 + c_3 \hat{E}^2 + \ldots \right) - \gamma_d \hat{E} \]
Mode evolution equation - sign of cubic nonlinearity

- First term leads to exponential growth, we must have a negative second term to have saturation.

\[
\frac{dA}{d\tau} = A(\tau - \frac{1}{2})^{\frac{\tau}{2}} d\tau \quad z^2 A(\tau - z) \int_0^{\tau - 2z} dx e^{-\hat{\nu}^3 z^2 (2z/3 + x) - \hat{\beta}(2z + x)} \times \nonumber \]
\[
A(\tau - z - x) A^*(\tau - 2z - x) \nonumber \]

\(\hat{\nu}\) - Diffusion coefficient

\(\hat{\beta}\) - Krook coefficient

- Minus sign must persist for steady state
- For Krook and diffusion – Sign can only flip for low collisionality

Berk et.al PRL, 76, 1256 (1996)
Breizman et.al PoP, 4, 1559 (1997)
Collisionality affects mode saturation

\[ \hat{v} = \frac{v}{\gamma_L - \gamma_d} \]

Heeter et al. PRL, 85, 3177 (2000)
Collisionality affects mode saturation

\[ \hat{\nu} = \frac{\nu}{\gamma_L - \gamma_d} \]

Heeter et al. PRL, 85, 3177 (2000)
Collisionality affects mode saturation

\[ \hat{\nu} = \frac{\nu}{\gamma_L - \gamma_d} \]

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Collisionality affects mode saturation

\[ \hat{\nu} = \frac{\nu}{\gamma_L - \gamma_d} \]
Collisionality affects mode saturation

\[ \hat{v} = \frac{v}{\gamma_L - \gamma_d} \]

Heeter et al. PRL, 85, 3177 (2000)
Collisionality affects mode saturation

- This was only a perturbative analysis (cubic order in $E$)
- Fully non-linear treatment requires numerical techniques
- Techniques should take advantage of separation of times scales $\left( \gamma \ll \omega \right)$ i.e. Use BOT code: Fourier space code that runs in a couple of minutes on a laptop
- What happens in the explosive regime?

\[(\gamma \ll \omega)\]
BREAK!
Marginal stability – No saturation

\[ \frac{\gamma_d}{\gamma_L} = 0.9 \]

Frequency is changing for \( \frac{\gamma_d}{\gamma_L} > 0.4 \)
Marginal stability – Frequency chirping

\[ \delta \omega \sim \sqrt{t} \]

\[ \omega_B \sim \text{const.} \]
Marginal stability – Frequency chirping

\[ \delta \omega \sim \sqrt{t} \]

\[ \omega_B \sim \text{const.} \]

\[ \beta = \nu = \alpha = 0.0, \gamma_d / \gamma_L = 0.900, 10 \text{ Harm, 20.0 box, } dt \times \gamma_L = 0.013, 3001 \text{ s points, } dt/ds = 1 \]
Marginal stability – Holes and clumps

Spectral lines are holes and clumps in phase space
Marginal stability – Hole/clump dynamics

- Holes/clumps are the original resonant particles
- They are modulated beams/anti-beams → large effect even with small density, since $\omega = kV_b$

$$\varepsilon \approx 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pb}^2}{(\omega - kV_{bc})^2} + \frac{\omega_{pb}^2}{(\omega - kV_{bh})^2} = 0$$

- Set $\omega = \omega_{pe}$ to lowest order. Correction from hole/clump will force other wave frequencies down/up → holes/clumps naturally move apart
**Marginal stability – Hole/clump dynamics**

- Holes/clumps are the original resonant particles
- They are modulated beams/anti-beams → large effect even with small density, since $\omega = kV_b$

\[
\frac{2\delta\omega}{\omega_{pe}} = \frac{\omega_{pb}^2}{(\omega_{pe} - kV_{bc})^2} - \frac{\omega_{pb}^2}{(\omega_{pe} - kV_{bh})^2}
\]

- Set $\omega = \omega_{pe}$ to lowest order. Correction from hole/clump will force other wave frequencies down/up → holes/clumps naturally move apart

\[\delta F \sim F'_0 \delta v\]
Marginal stability – Hole/clump dynamics

- They move slowly compared to the bounce period
- Particles can't get inside separatrix $\rightarrow$ waterbag
- Trapped particles give most of $\delta n_e$

$$\nabla \cdot \varepsilon E = -|e| \delta n_e \rightarrow \delta \omega \omega_B^2 = \gamma L \delta \omega \omega_B$$

- Works for $\omega \approx \omega_{pe}$
- $\rightarrow$ $E$ is constant
- Hole or clump gets deeper/higher as it moves
Marginal stability – Hole/clump dynamics

- Hole: energy is required to move particles up \( \sim F_{\text{in}} \)
- Energy released as particles are forced over \( \sim F_{\text{out}} \)
- They must move to balance dissipation
- Deeper holes release more energy \( \rightarrow \sqrt{t} \) chirp

\[
\gamma_d \omega_B^4 = \gamma_L \delta \omega \omega_B \frac{\partial \delta \omega}{\partial t}
\]

Dissipation power release
The Questions

- How does a low density population produce a large effect
- How does the plasma produce such rich non linear evolution at different timescales
- How is it that the same modes driven by different particles look so different

ICRH drive (JET)  
NBI drive (MAST)

Heeter et.al PRL 85, 3177 (2000)
Pinches et.al PPCF, 46, S47 (2004)
Collisionality - Revisited

- Collisionality not low enough to explain MAST
- NBI distribution determined by drag for $E \sim E_A >> E_{crit}$
- Dynamical friction (drag) collisions should be included

$$\left. \frac{dF}{dt} \right|_{coll} = \alpha^2 \left( \frac{\partial F}{\partial \nu} - \frac{\partial F_0}{\partial \nu} \right)$$

- Could this explain the bursting for beam driven TAEs
Mode evolution equation – Effect of drag

Near marginal stability the amplitude ($A$) of the unstable mode evolves according to the following equation:

$$\frac{dA}{d\tau} = A(\tau) - \left(\frac{1}{2}\right)^{\tau/2} dz z^2 A(\tau - z) \int_0^{\tau-2z} dx e^{-\hat{\nu}^2 z^2 (2z/3+x) - \hat{\beta}(2z+x) + i\hat{\alpha}^2 z(x+z)} \times A(\tau - z - x) A^*(\tau - 2z - x)$$

- $\hat{\nu}$ - Diffusion coefficient
- $\hat{\beta}$ - Krook coefficient
- $\hat{\alpha}$ - Drag coefficient

• Drag gives oscillatory behaviour, in contrast to the Krook and diffusive cases.
• For drag – The oscillatory nature allows the sign to flip often → don't need low collisionality to get explosion.
Marginal stability - Diffusion + drag

• For diffusion drag steady state solutions do exist
• For an appreciable amount of drag these solutions become unstable (pitch fork splitting etc.)
• Explosive solutions again when drag dominates

Lilley et.al PRL, 102, 195003 (2009)
Fully nonlinear drag regime - expectations

- Drag provides a preferred direction
- Expect asymmetry
- Holes move up in velocity
- Drag provides a flow, this acts like chirping
- Can drag replace chirping?
- i.e. Can we get a steady state non-linear state away from the original resonance?
Pure drag

![Graph showing stability and stability state regions.](graph.png)

- **Stable Steady State**
- **No Steady State**

Variables indicated on the x and y axes.
Pure Drag – Holes Grow Faster, Clumps Decay

\[ \alpha/(\gamma_L - \gamma_d) = 1.5, \frac{\gamma_d}{\gamma_L} = 0.9 \]

\[ \omega/\gamma_L \]}

\[ t \times \gamma_L \]}

\[ (k\nu - \omega)/\gamma_L \]

\[ (k^2 - \omega^2)/\gamma_L \]}

Pure drag – Growing holes

• Drag collision operator has a slowing down force and a sink

\[ \frac{dF}{dt} \bigg|_{\text{coll}} = \frac{\alpha^2}{k} \left( \frac{\partial F}{\partial v} - \frac{\partial F_0}{\partial v} \right) \]

• Slowing down + sink returns distribution to equilibrium

• E field however can hold the hole in place working against slowing down force

• Sink still acts to lower F \( \rightarrow \) deeper hole over time
Pure drag – Growing holes

- Deeper hole $\rightarrow$ bigger $E$

$$\delta \omega \omega_B^2 = \gamma_L \omega_B g$$

$$g = \delta \omega + \alpha^2 t$$

$$\gamma_d \omega_B^4 = \gamma_L g \omega_B \left( \frac{\partial \delta \omega}{\partial t} + \alpha^2 \right)$$

- When drag dominates no steady state is possible

$$\omega_B \sim \alpha^{4/3} t^{1/3}, \quad \delta \omega \sim \gamma_L (\alpha t)^{2/3}$$
Pure drag – Saturation without slope

Remove slope and saturation is achieved
Now add some diffusion
Drag + diffusion – Steady state hole

\( \frac{\omega}{\gamma_L} = 1.00 \), \( \alpha/(\gamma_L - \gamma_d) = 1.50 \), \( \gamma_d / \gamma_L = 0.900 \), 10 Harm, 10.0 box, \( dt \times \gamma_L = 0.020 \), 1001 s points, etc.
Add a bit more diffusion
Drag + diffusion – Undulating frequency

\( \nu/(\gamma_L - \gamma_d) = 1.02, \, \alpha/(\gamma_L - \gamma_d) = 1.50, \, \gamma_d/\gamma_L = 0.9, \) 10 Harm, 10 box, \( dt \times \gamma_L = 0.020 \), 1001 s points, \( dt/dx \)
Keep adding diffusion

- Stable steady state
- No steady state
Drag + diffusion – Hooked frequency chirp

- Hooked frequency chirp seen in BOT
- Also seen in MAST (NBI) and JET (ICRH)
Drag + diffusion – Hooked frequency chirp

Hooks for the holes, clumps die sooner

Lilley et.al PoP, 17, 092305 (2010)
Drag Diffusion competition

\[ \delta \omega B^2 = \gamma_L \omega_B g \]

\[ \frac{\partial g}{\partial t} + \frac{\nu^3}{\omega_B^2} g = \frac{\partial \delta \omega}{\partial t} + \alpha^2 \]

\[ \gamma_d \omega_B^4 = \gamma_L g \omega_B \left( \frac{\partial \delta \omega}{\partial t} + \alpha^2 \right) \]

- Poisson Equation
- Diffusion fills, chirping and drag deepen
- Energy balance
0-D Equations

\[ x^2 = y \left( \frac{\partial y}{\partial \tau} + 1 \right) \]

\[ a \frac{\partial (xy)}{\partial \tau} + \frac{y}{x} = \left( 1 + \frac{\partial y}{\partial \tau} \right) \]

- x=y=1 is steady state
- Unstable for a<1
- Stable for a>1

Generalisation to toroidal systems
Toroidal systems – A first glance (low freq.)

- Phase space resonance is more sophisticated

\[ u \equiv k \nu - \omega = 0 \quad \Rightarrow \quad \Omega \equiv n \omega \varphi \left( P_\varphi, E \right) - p \omega \theta \left( P_\varphi, E \right) - \omega = 0 \]

- Location of resonance varies

![Graph showing resonance and wave motion]

\[ E - \frac{\omega}{n} p_\varphi = \text{const.} \]

Breizman et.al PoP, 4, 1559 (1997)
Toroidal systems – Reduction to 1-D model (low freq.)

- Particle motion along resonance does not lead to strange gradients in $F$, so neglect them
- Need projection of motion and collisions across resonance

\[ E - \frac{\omega}{n} p_\phi = \text{const.} \]

Breizman et.al PoP, 4, 1559 (1997)
Toroidal systems – Reduction to 1-D model (low freq.)

- Transform to coordinates that straighten resonance
- Motion across the resonance 1-D for given $E$ and $\mu$
- Must integrate over all $E$ and $\mu$ to get the result

\[
\Omega \equiv n\omega_\phi(P_\phi, E) - p\omega_\theta(P_\phi, E) - \omega = 0
\]

Breizman et.al PoP, 4, 1559 (1997)
The Questions

- How does a low density population produce a large effect
- How does the plasma produce such rich non linear evolution at different timescales
- How is it that the same modes driven by different particles look so different

ICRH drive (JET)  

NBI drive (MAST)

Heeter et.al PRL 85, 3177 (2000)  
Pinches et.al PPCF, 46, S47 (2004)
Experimental estimate for MAST and ITER

\[
\frac{\nu_{\text{TAE}}}{\alpha_{\text{TAE}}} \approx \frac{T_e^{3/4} n_e^{1/6}}{B_0^{5/6}}
\]

Drag vs diffusion depends on plasma parameters

\[
\frac{\nu_{\text{TAE}}}{\alpha_{\text{TAE}}} \approx 0.2 - 1.6
\]

MAST - beams - Drag can dominate \(\rightarrow\) explosive

\[
\frac{\nu_{\text{TAE}}}{\alpha_{\text{TAE}}} \approx 1.4
\]

ITER – alphas – drag and diffusion comparable

Lilley et al. PRL, 102, 195003 (2009)
Remaining tasks...not exhaustive!

- 1D model: Can we understand more about why marginal stability seeds holes & clumps
- 1D model: Extending to high frequency involving cyclotron resonance
- 3D world: Put drag into fully toroidal codes to look at TAEs – now being done in HAGIS
- 3D world: Experimentally scan parameter space – needed to predict ITER operation
Conclusions

- Waves are important
- Resonance empowers the fast particles
- Marginal stability produces surprising route to energetic particle modes
- Plenty of non-linear scenarios enriched by collisions
- Drag provides destabilising effect and gives an important observed asymmetry
- 1D model is good: for single resonance only
BOT Code

• Email bumpontail@gmail.com from your work email
• Please give your name, institution and your position
• It is free for you to use, modify and also distribute, but I encourage others to contact me for the code so that I can send updates as they become available