Extension of conventional MHD equilibrium theory to model the fast particle effects

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Motion of a single particle

\[ m_p \frac{dv_p}{dt} = F = q_p (E + v_p \times B) \]

depends on the electric and magnetic fields \( E \) and \( B \) created by all other particles and external sources.

\[ B(r) = \frac{\mu_0}{4\pi} \int \mathbf{j}(r') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dV' \]
In theory of tokamaks and stellarators, the bulk plasma is most frequently considered as a continuous medium described by the single-fluid MHD equations.

Is it always good? We consider some other options.
Standard MHD equations

**Force balance:**  \( \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mathbf{j} \times \mathbf{B} \)

with

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla
\]

\( \nabla \cdot \mathbf{B} = 0 \), \( \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \), \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)

\& sometimes \( \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \)  \( \Rightarrow \) magnetic flux conservation

⇒ in equilibrium \( \nabla p = \mathbf{j} \times \mathbf{B} \)
Currents in the equilibrium plasma

With \( \nabla p = j \times B \) in equilibrium, we have,

\[
\nabla \times B = 0 = \nabla \cdot j
\]

\[
\nabla \cdot j = 0 \implies \nabla \cdot j_{\parallel} = -\nabla \cdot j_{\perp} = \frac{B \times \nabla p}{B^2} \cdot \nabla B^2
\]

Find \( j_{\parallel} \) and solve \( \nabla \times B = \mu_0 j \) (with \( \nabla \cdot B = 0 \))
Alternative: kinetic approach

Boltzmann eq:
\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m_p} \cdot \nabla_v f = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}
\]

when averaged:
\[
\frac{\partial}{\partial t} \rho \mathbf{v} + \nabla \cdot \mathbf{\Gamma} = \mathbf{f}
\]

with
\[
\mathbf{\Gamma} = \rho \mathbf{vv} - \mathbf{BB} + \frac{\mathbf{B}^2}{2} \mathbf{I} + \mathbf{\rho}
\]

and
\[
\rho = \rho_i \langle u_i u_i \rangle + \rho_e \langle u_e u_e \rangle
\]

5th ITER International Summer School, June 21, 2011
Distribution of fast ions produced by additional heating systems

“is strongly anisotropic,

with the NBI produced fast ions flowing predominantly parallel to the magnetic field, and the ICRH accelerated ions characterized by large perpendicular energy and mostly trapped orbits”

With such fast ions \( \vec{p} \neq p\vec{I} \) and \( \nabla \cdot \vec{p} \neq \nabla p \)

Then we assume

\[
\vec{p} = p_\parallel \frac{B B}{B^2} + p_\perp \left( \vec{I} - \frac{B B}{B^2} \right)
\]

the most simple form of the pressure tensor with anisotropy.

\[
(p_\parallel, p_\perp) = \sum m_p \int (v_\parallel^2, \frac{v_\perp^2}{2}) f d\vec{v}_p
\]

parallel and perpendicular pressures
From isotropic to anisotropic equil.

Instead of \( \nabla p = \mathbf{j} \times \mathbf{B} \) in equilibrium

we have \( \nabla \cdot \mathbf{p} = \mathbf{j} \times \mathbf{B} \) with

\[
\mathbf{p} = p_\parallel \frac{\mathbf{B} \mathbf{B}}{B^2} + p_\perp \left( \mathbf{I} - \frac{\mathbf{B} \mathbf{B}}{B^2} \right)
\]

There is also \( \mathbf{j} = \sum q_p \int v_p f dv_p \), but ...
With fast particles, $p_{\parallel}$ and $p_\perp$ can be different. What consequences?

To what extent $p_{\parallel} \neq p_\perp$?

How can we prescribe $p_{\parallel}$ and $p_\perp$?

Should we develop new theory?
Examples from Zwingmann et al 2001 *PPCF* 43 1441

<table>
<thead>
<tr>
<th>JET:</th>
<th>$p_{\parallel}$ and $a = \text{const}$</th>
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<td>Tore Supra:</td>
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**General relations**

Start from general equilibrium equations

\[
\nabla \cdot \vec{p} = \vec{j} \times \vec{B} , \quad \mu_0 \vec{j} = \nabla \times \vec{B} , \quad \nabla \cdot \vec{B} = 0 ,
\]

\[
\vec{p} = p_\parallel \frac{\vec{B} \cdot \vec{B}}{\vec{B}^2} + p_\perp \left( \vec{I} - \frac{\vec{B} \cdot \vec{B}}{\vec{B}^2} \right).
\]

As a result we have

\[
\nabla p_\parallel = \sigma_\parallel \nabla (\frac{\vec{B}^2}{2}) + \vec{K} \times \vec{B}
\]

with \( \mu_0 \vec{K} = \nabla \times (\sigma \vec{B}) \), \( \sigma = 1 - \sigma_\parallel \), and \( \sigma_\parallel = \frac{p_\parallel - p_\perp}{\vec{B}^2} \).
Most popular assumptions

\[ p_\parallel = p_\parallel(a, B), \quad p_\perp = p_\perp(a, B) \]

with \( a = \text{const} \) the flux coordinate: \( B \cdot \nabla a = 0 \)

1. Good for symmetry (tokamaks),

2. Corresponds to the leading order solution of the Fokker–Planck equation for the distribution function \( f \) (which is \( B \cdot \nabla f = 0 \) in this case)

Other models? Better choice of \( p_\parallel \) and \( p_\perp \)?
Examples of $p_\parallel$ and $p_\perp$ prescription


$$P'_\parallel = p'_i(\bar{\Psi}) + p'_a(\bar{\Psi}, R) = \sum_{k=1}^{NP} c_k g_k(\bar{\Psi}; 1) + \sum_{k=1}^{NP} \sum_{n=1}^{NA} c_{k+n} f_n(\vec{r}) g_k(\bar{\Psi}; \delta).$$

$$P_\perp(\Psi, R) = P_\parallel(\Psi, R) + R \frac{\partial P_\parallel(\Psi, R)}{\partial R}.$$  

“The present analysis was carried out with one anisotropy term”

“contributions from neutral beams and/or RF heating are obtained from suitable power deposition codes”
Examples of $p_{\parallel}$ and $p_{\perp}$ prescription


\[
F(s, E, \mu) = \frac{h(s)}{E^{3/2} + E_c^{3/2}} \left[ 1 - \frac{\mu B_m(s)}{E} \right]^L.
\]

\[
p_{b\perp}(s, B) = p_{b\parallel}(s, B) - B \frac{\partial p_{b\parallel}}{\partial B} \bigg|_s.
\]

“modified slowing down distribution”

“model the effects of balanced tangential neutral beam injection”
Examples of $p_\parallel$ and $p_\perp$ prescription

Cooper W A et al 2006 Anisotropic pressure bi-Maxwellian distribution function model for three-dimensional equilibria Nucl. Fusion 46 683

$$\mathcal{F}_h(s, \varepsilon, \mu) = \mathcal{N}(s) \left( \frac{m_h}{2\pi T_\perp(s)} \right)^{3/2} \times \exp \left[ -m_h \left( \frac{\mu B_C}{T_\perp(s)} + \frac{|\varepsilon - \mu B_C|}{T_\parallel(s)} \right) \right].$$

$$p_\perp(s, B) = p_\parallel(s, B) - B \frac{\partial p_\parallel}{\partial B} \bigg|_s$$

“Large parallel and perpendicular anisotropy factors can be explored through the choice of the temperature ratio $T_\parallel/T_\perp$”
Examples: Contours of constant $f$

**model for balanced tangential NBI**


Cooper W A *et al* 2006 *Nucl. Fusion* 46 683

(Bi-Maxwellian) $\Rightarrow$

**function with large parallel anisotropy**

$T_\perp/T_\parallel=0.35$

$B/\parallel=0.8$

$T_\perp/T_\parallel=0.35$

$B/\parallel=1.6$
Parallel force balance

\[ \nabla p_\parallel = \sigma_\parallel \nabla \left( \frac{B^2}{2} \right) + K \times B \]

\[ \Rightarrow B \cdot \nabla p_\parallel = \sigma_\parallel B \cdot \nabla \left( \frac{B^2}{2} \right) \]

which is equivalent to

\[ B \cdot \nabla (p_\parallel + p_\perp) = -B^2 B \cdot \nabla \sigma_\parallel \]

We have

\[ B^2 = B_0^2 + (B^2 - B_0^2) \]

with \( \left| \frac{B^2}{B_0^2} - 1 \right| \ll 1 \) in tokamaks and stellarators. Then

\[ p_\parallel + p_\perp + B_0^2 \sigma_\parallel = C(a). \]

\[ p_\parallel \left( 1 + \frac{B_0^2}{B^2} \right) + p_\perp \left( 1 - \frac{B_0^2}{B^2} \right) = 2p_\parallel_0 + \delta \]
Parallel force balance: consequences

\[ p_{\parallel} \approx p_{\parallel 0} + \frac{p_{\parallel 0} - p_\perp}{2} \left(1 - \frac{B_0^2}{B^2}\right) \]

with \( p_{\parallel 0} = p_{\parallel 0}(a) \)

\[ p_{\parallel} = p_{\parallel 0} + \tilde{p}_{\parallel} \]

Large \( \tilde{p}_{\parallel} \) can be produced by very large \( \tilde{p}_\perp \) only.

⇒ in tokamaks and stellarators, \( p_{\parallel} - p_{\parallel 0}(a) \)

must be small even at large variations of \( p_\perp \).
Some numerical results


“the total pressure surfaces with $p_\parallel >> p_\perp$ do not appear to significantly deviate from the flux surfaces which is in stark contrast to earlier results with $p_\perp >> p_\parallel$ where the pressure surfaces can become completely decoupled from the flux surfaces”


“Significant differences between parallel and perpendicular pressure anisotropy are observed.”

“poloidal variation in $p_\parallel$ is only non-negligible when $p_\perp >> p_\parallel$”
Examples from Zwingmann et al 2001 *PPCF* 43 1441

**JET:** \( p \parallel \) and \( a = \text{const} \)

**Tore Supra:** \( p \perp \) and \( a = \text{const} \)
Perpendicular force balance

\[ \nabla p_{||} = \sigma_{||} \nabla (B^2 / 2) + K \times B \]

with \( \mu_0 K = \nabla \times (\sigma B) \) \( \Rightarrow \)

\[ j_\perp = \frac{B}{\sigma B^2} \times \left( \nabla p_\perp + \frac{p_{||} - p_\perp}{B^2} \nabla \frac{B^2}{2} \right) \], mainly determined by \( p_\perp \).

After some algebra (cylinder):

\[ 2 \frac{\Delta \Phi}{\Phi_0} = \frac{B_J^2}{B_0^2} - \bar{\beta}_\perp + 2 \frac{\Delta \Phi}{\Phi_0} \]

where \( \Delta \Phi = \int_{S_\perp} (B - B_v) \, dS_\perp \) is the diamagnetic signal.
Equilibrium current, \textit{general}

\[ \nabla \cdot K_{||} = -\nabla \cdot K_{\perp} = \frac{\mathbf{B} \times \nabla (p_{\parallel} + p_{\perp})}{2\sigma B^4} \cdot \nabla (B^2 + 2p_{\perp}) \]

with \( \sigma \mathbf{j} = K + \nabla \sigma \times \mathbf{B} / \mu_0 \) and \( \sigma = 1 - (p_{\parallel} - p_{\perp}) / B^2 \approx 1 \)

If \( \nabla (B^2 + 2p_{\perp}) \) could be replaced by \( \nabla B^2 \), we would obtain \( K \) (and \( \mathbf{j} \)) depending on \( p_{\parallel} + p_{\perp} \).

Therefore, \( p_{\perp} \) is a key function.
Equilibrium current, simplified, $j_{||}$

$$\nabla \cdot K_{||} = -\nabla \cdot K_{\perp} = \frac{B \times \nabla (p_{||} + p_{\perp})}{2 \sigma B^4} \cdot \nabla (B^2 + 2p_{\perp})$$

With $p_{||} \approx p_{||0}$ and $|\tilde{p}_{\perp}| \ll \varepsilon B^2$, we have

$$\nabla \cdot j_{||} \approx \frac{B \times \nabla (p_{||} + p_{\perp})}{2 \sigma B^4} \cdot \nabla B^2.$$
Equil. currents, simplified, summary

\[ \nabla \cdot \vec{\jmath} = \vec{j} \times \mathbf{B} \quad \text{with} \quad \vec{\jmath} = \vec{p}_\parallel \frac{\mathbf{B} \cdot \mathbf{B}}{\mathbf{B}^2} + \vec{p}_\perp \left( \mathbf{i} - \frac{\mathbf{B} \cdot \mathbf{B}}{\mathbf{B}^2} \right). \]

Perpendicular:
\[ \vec{j}_\perp \approx \frac{\mathbf{B} \times \nabla \vec{p}_\perp}{\mathbf{B}^2}, \quad \text{determined by} \quad \vec{p}_\perp \]

Parallel:
\[ \nabla \cdot \vec{j}_\parallel \approx \frac{\mathbf{B} \times \nabla (\vec{p}_\parallel + \vec{p}_\perp)}{2 \mathbf{B}^4}. \quad \nabla \mathbf{B}^2, \quad \text{determined by} \quad \vec{p}_\parallel + \vec{p}_\perp. \]
Poloidal $\psi$ and toroidal $\Phi$ magnetic fluxes associated with a toroidal magnetic surface

$$\psi \equiv \int B \cdot dS_{pol}$$

$$F \equiv \int j \cdot dS_{pol}$$

$$\Phi \equiv \int B \cdot dS_{tor}$$

$$2\pi B = \nabla \psi \times \nabla \zeta + F \nabla \zeta$$
Magnetic diagnostics

\( \psi \) is determined by \( j_\parallel \),
\[ \Rightarrow \text{ by } p_\parallel + p_\perp \]

\( \Phi \) is determined by \( j_\perp \),
\[ \Rightarrow \text{ by } p_\perp \]
“In low density discharges of a Large Helical Device (LHD), anisotropic pressure is expected because the LHD has powerful tangential neutral beam injection systems. We show the strong correlation between the pressure anisotropy due to the beam pressure based on Monte Carlo calculations and the ratio of the diamagnetic loop signal and the saddle loop signal.”
Large Helical Device (LHD)

All superconducting coil system

Major radius = 3.42 - 4.1 m

Plasma radius = 0.6 m

Plasma volume = 30 m$^3$

Toroidal field 2.9 T

(S. Sudo, 2003)
Anisotropic Pressure Effect on the MHD Equilibrium in LHD

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Numerical calculated prediction of anisotropic pressure from the beam


The birth profile of fast ion from NBI is estimated by Monte-Carlo simulation.

Beam pressure is estimated by the steady state solution of the Fokker-Planck eq.

Direct loss effect is taken into account.

\[ W_{||} = \frac{1}{3} W_{\text{thermal}} + W_{\text{beam\|}} \]
\[ W_{\perp} = \frac{2}{3} W_{\text{thermal}} + W_{\text{beam\perp}} \]

\[ (W_{\text{dia}} = W_{\text{thermal}} + \frac{3}{2} W_{\text{beam\perp}}) \]

\[ \frac{T_{i}}{T_{e}} \quad \frac{W_{\perp}}{W_{\|}} \]

\[ \Phi_{\text{Pexp}/\Phi_{\text{Piso}}} \]

\[ n_{e}[10^{19} \text{ m}^{-3}] \]

Consistent

Numerical calculated prediction

Estimation by Magnetics
Plasma equilibrium with toroidal rotation

\[ \rho (v \nabla) v = - \nabla \hat{p} + j \times B \]

Scalar pressure:

\[ \frac{\rho v_i^2}{r} e_r - \nabla p + j \times B = 0 \]

Additional force along the major radius

\[ \int \frac{p + \rho v_i^2}{r} dV + \int e_r (j \times B) dV = 0 \]
Equilibrium with toroidal rotation, estimates

\[
\frac{p}{\rho} = v_{T_i}^2 \left( 1 + \frac{n_e T_e}{n_i T_i} \right)
\]

with

\[
\rho v_{t}^2
\]

\[
\frac{p}{\rho} \approx \frac{v_{t}^2}{2 v_{T_i}^2}
\]

For hydrogen

\[
v_{T_i} = 979 \sqrt{\frac{T_i}{T_0}} \text{ km/s}
\]

with

\[
T_0 = 10 \text{ keV}
\]

Large

\[
-\rho (\mathbf{v} \nabla) \mathbf{v} = e_r \rho v_{i}^2 / r
\]

at very large velocity only
Plasma with toroidal rotation, estimates

Proton mass

\[ m_p = 1.67 \times 10^{-27} \text{ kg} \]

Plasma density

\[ n = 10^{20} \text{ m}^{-3} \]

⇒ mass density

\[ \rho = m_p n = 1.67 \times 10^{-7} \text{ kg/m}^3 \]

Compare to

water \[ \rho = 10^3 \text{ kg/m}^3 \]

air \[ \rho = 1.29 \text{ kg/m}^3 \]

ITER: plasma volume

\[ V_{\text{plasma}} = 870 \text{ m}^3 \]

Mass of H plasma

\[ M = \rho V_{\text{plasma}} = 1.45 \times 10^{-4} \text{ kg} \]
Rotation and Shafranov shift

\[ \Delta' = \Delta'_S - \frac{a}{R} \frac{\rho v_t^2 - \rho v_i^2}{B^2_{\theta}} \]

with

\[ \Delta'_S = -\frac{a}{R} \left[ \frac{l_i}{2} + 2 \frac{\rho - \rho}{B^2_{\theta}} \right] \]

with \[ l_i \equiv \frac{B^2_{\theta}}{B^2_{\theta}} \]

and

\[ \bar{X} \equiv \frac{2}{a^2} \int_0^a X \rho d\rho \]
Rotation and Shafranov shift - 2

\[ \Delta'(b) = -\frac{b}{R} \left[ \frac{l_i}{2} + \frac{2p + \rho v_t^2}{B_J^2} \right] \]

The global effect of toroidal rotation is larger outward shift, but only weak increase

Effect comparable to pressure at

\[ v_t \sim v_{T_i} \]

or

\[ v_{beam} \sim v_{T_i} \frac{S_{\text{plasma}}}{S_{\text{beam}}} \]

for a beam
Summary

Fast particles create the pressure anisotropy and rotation

In equilibrium, the deviations from conventional MHD must be mainly related to $p_\perp$

\[ p_{\parallel}\left(1 + \frac{B_0^2}{B^2}\right) + p_{\perp}\left(1 - \frac{B_0^2}{B^2}\right) \approx 2p_{\parallel0} \]

In some cases it must be possible to estimate the degree of pressure anisotropy by magnetic measurements

$\mathbf{j}_\perp$ is determined by $p_{\perp}$, while $\mathbf{j}_\parallel$ is $\sim$ determined by $p_{\parallel} + p_{\perp}$

Reliable when $p_{\perp} << p_{\parallel}$ or $p_{\perp} \approx p_{\perp0}(\alpha)$
Toroidal rotation gives slightly larger Shafranov shift, but a strong effect is only at very large speed.

For more details see


and references therein
Backup slides
Experiments on T-10


Electron temperature profiles for several time slices after on-axis ECRH switching on (shot #32916, input on-axis ECRH power ~600 kW)

“in the heating region plasma can ‘assimilate’ only part of the input power”

“up to 60% of ECRH power is rapidly thrown out of the plasma core to the peripheral plasma (‘ballistic effect’)”

“effective heat diffusivity increases up to values of 10–15 m² s⁻¹ in the first 100–200 μs and decreases down to values of 1.5–2.0 m² s⁻¹ during the following 1–2 ms.”
Experiments on TEXTOR


“the absorbed energy is perfectly confined inside the q=1 surface during the first 5 ms”

“the electron heating rate inside the q=1 surface calculated from the local TS data shows ~200 kW which is only one third of the launched EC power”

\[
\frac{3}{2} \frac{\partial}{\partial t} (nT) = -\frac{1}{r} \frac{\partial}{\partial r} rW + P_{OH} + Q + P_{EC}
\]

\[W = -\chi_e \nabla T + nT \vec{u}\]

\(P_{OH}\) the Ohmic power, \(Q\) other heat sources, \(P_{EC}\) the ECRH power. Later \(\vec{u}\) is disregarded.

“the relative density variation is much less than the relative temperature variation”

No electromagnetic interaction here
Our main equations in


**Force balance:** \( \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} \quad \Rightarrow \quad \nabla p = \mathbf{j} \times \mathbf{B} \)

**Energy balance:** \( \frac{\partial}{\partial t} \left( \frac{3}{2} p + \frac{\mathbf{B}^2}{2} \right) + \nabla \cdot \left( \frac{5}{2} p \mathbf{v} + \mathbf{E} \times \mathbf{B} + q_1 \right) = s \)

**Maxwell eqns:** \( \nabla \cdot \mathbf{B} = 0 \), \( \nabla \times \mathbf{B} = \mathbf{j} \), \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \)

\( \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \quad \Rightarrow \quad \text{magnetic flux conservation:} \)

\( \Phi_{pl} \equiv \int_{\text{plasma}} \mathbf{B} \cdot d\mathbf{S} = \text{inv} \)

\( \Phi_e \equiv \int_{\text{gap}} \mathbf{B} \cdot d\mathbf{S} = \text{inv} \)

plasma, plasma-wall gap
Integral energy balance

Integrate

\[
\frac{\partial}{\partial t}\left(\frac{3}{2} p + \frac{B^2}{2}\right) + \nabla \cdot \left(\frac{5}{2} p\mathbf{v} + \mathbf{E} \times \mathbf{B} + \mathbf{q}_l\right) = s
\]

up to the wall (\(\mathbf{E} \times \mathbf{n} = 0\)):

\[
\frac{d}{dt} \int \frac{3}{2} p dV = \int p_{in} dV - \frac{d}{dt} \int \frac{B^2}{2} dV
\]

last term - “missing power”

The magnetic energy change is small,

\[
\delta W_m^{pl} + \delta W_m^{gap} \approx 0
\]

The heating power goes to the plasma,

no “missing power” at fast processes.