Non diffusive transport in fusion plasmas: a fractional diffusion approach

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Diffusive transport

•Consider the transport of a single scalar field in one dimension

$$\partial_t T = -\partial_x q + S$$

•According to the standard diffusive paradigm

$$q_d = -D\,\partial_x T + V\,T$$

Diffusion coefficient $D = D(x,t;T,\partial_x T)$

Velocity pinch $V = V(x,t)$

$$\partial_t T + \underbrace{\partial_x[V\,T]}_{\text{Convective transport}} = \underbrace{\partial_x[D\partial_x T]}_{\text{Diffusive transport}} + S$$

•Within this framework, the goal of transport modeling is to find V and D based on theory, numerics and experimental evidence.

•This approach has been quite useful and valuable for the understanding of transport in fusion plasmas.

However……

Non-diffusive transport

- The "V-D" paradigm is a local description that assumes a well-defined transport scale and that widely separated regions do not interact.

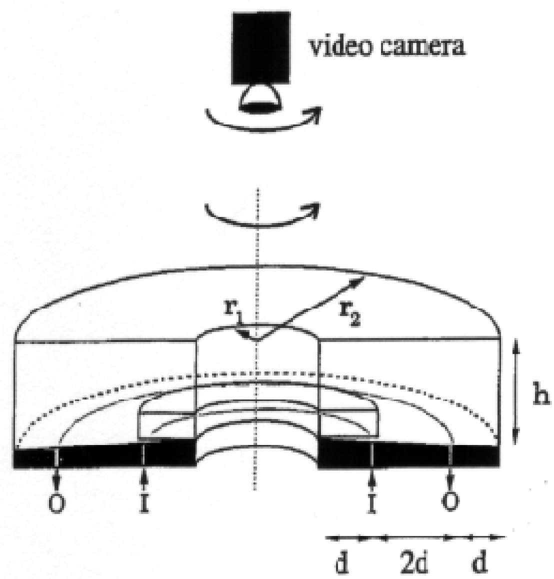
- There is evidence that indicates that this assumption might be too restrictive. Some examples include:

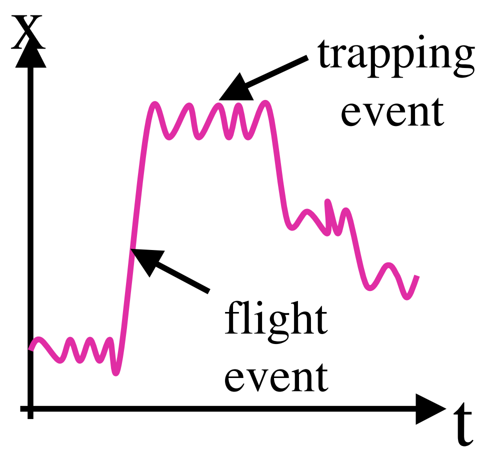
  - Non-diffusive scaling, and non-Gaussian statistics
  - Fast propagation phenomena, and non-local transport
  - Up-hill transport, density peaking, pinch effects, ……

- An important current problem in fusion research is to develop models capable of describing these non-diffusive transport phenomena.

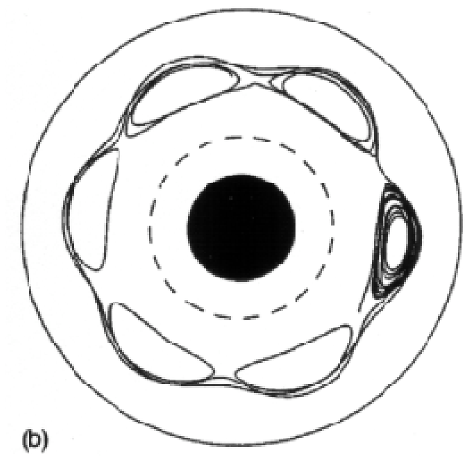
- We present an overview on non-diffusive transport, and discuss a recently proposed framework to model this type of transport.

Example 1: passive tracers in fluids

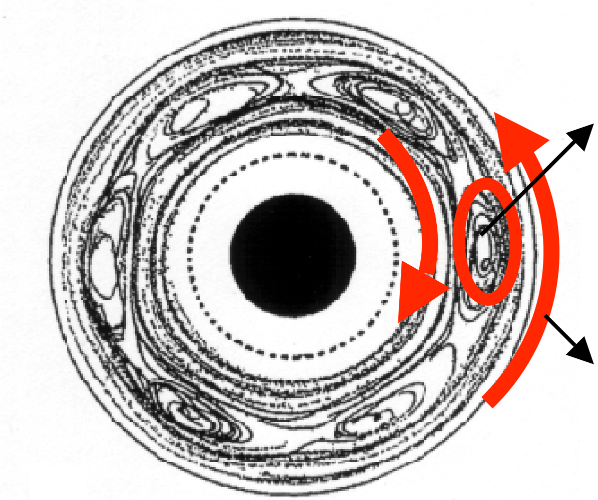




Model



Experiment

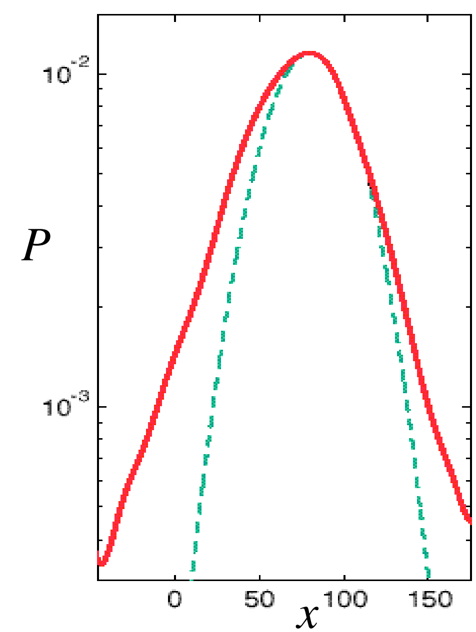


Vortices induce particle trapping

Zonal flows induce particle "flights"

Super-diffusive scaling

$$\langle \delta r^2 \rangle \sim t^{4/3}$$

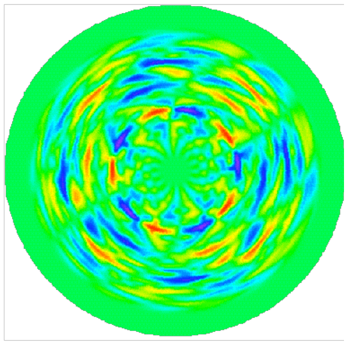


Solomon et al, Phys. Rev. Lett. **71,** 3975 (1993)

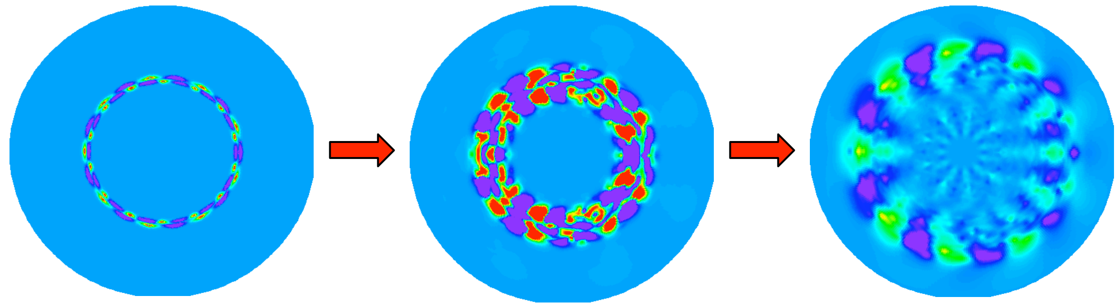
D. del-Castillo-Negrete, Phys. Fluids **10,** 576 (1998)

Example 2: passive tracers in plasma turbulence

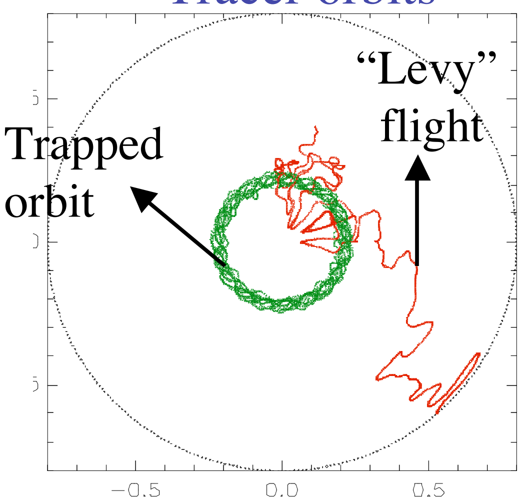
E*x*B flow velocity eddies induce particle trapping



"Avalanche like" phenomena induce flights that lead to spatial non-locality



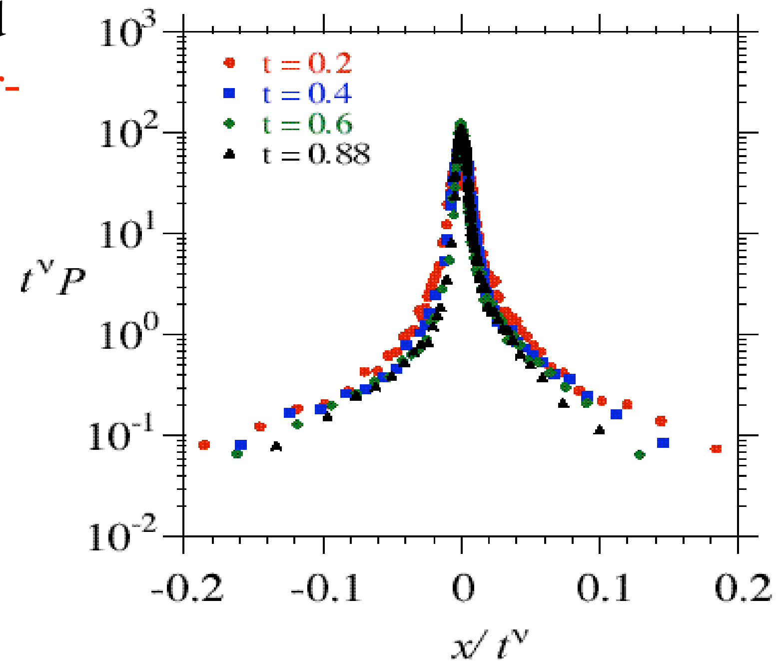
Tracer orbits



Particle trapping and flights leads to super-diffusive scaling

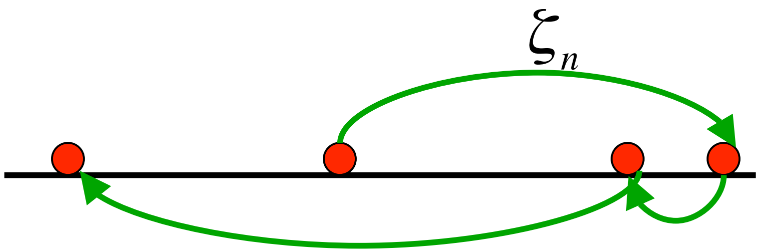
$$\left\langle \delta r^n \right\rangle \sim t^{2n/3}$$

Non-Gaussian (Levy) distribution



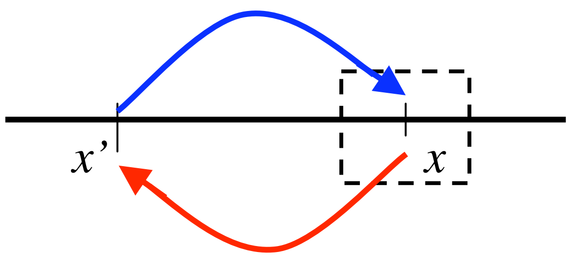
D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004)

"Microscopic" description of diffusion equation: the Brownian random walk



ζ_n = jump $\qquad \lambda(\zeta)$ = jump size pdf

$\langle \zeta^2 \rangle$ = characteristic transport scale



Master equation

$$\partial_t P = \int_{-\infty}^{\infty} dx' \left[\underbrace{\lambda(x-x')P(x',t)}_{\text{gain}} - \underbrace{\lambda(x-x')P(x,t)}_{\text{loss}} \right]$$

$$\partial_t P = \int_{-\infty}^{\infty} dx' \, \lambda(x') P(x-x',t) - P(x,t)$$

Localization assumption

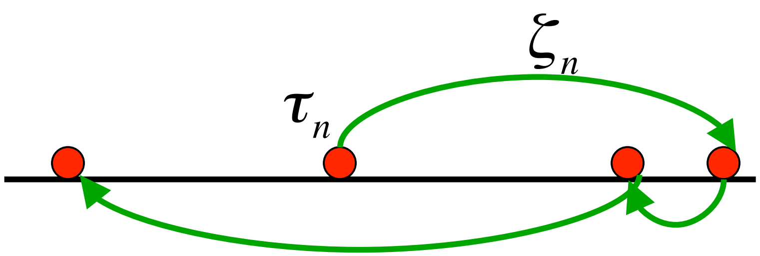
$$= \int_{-\infty}^{\infty} dx' \, \lambda(x') \left[P - x' \partial_x P + \frac{1}{2} x'^2 \, \partial^2{}_x P + \cdots \right] - P(x,t)$$

$$\langle x \rangle = V = \text{Pinch}$$

$$\frac{1}{2} \langle x^2 \rangle = D = \text{Diffusion}$$

$$\partial_t P = -V \partial_x P + D \, \partial^2{}_x P$$

Revisiting the foundation of the diffusion equation: the Continuous Time Random Walk



τ_n = waiting time $\quad \psi(\tau)$ = waiting time pdf

ζ_n = jump $\quad \lambda(\zeta)$ = jump size pdf

$\tilde{\phi}(s) = s\tilde{\psi}/(1-\tilde{\psi})$ $\quad \phi(\tau)$ = memory function

<u>Montroll-Weiss aster equation</u>

$$\partial_t P = \int_0^t dt'\, \phi(t-t') \int_{-\infty}^{\infty} dx' \left[\lambda(x-x')P(x',t) - \lambda(x-x')P(x,t)\right]$$

<u>Solution in Fourier Laplace space</u>

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s} \frac{1}{1-\tilde{\psi}(s)\,\hat{\lambda}(k)}$$

(Montroll-Weiss 1965)

Fluid limit of the CTRW:
Derivation of fractional diffusion equation

Long-time, large-scale asymptotic limit of the Montroll-Weiss equation

$$\psi^*(t) = \frac{1}{\varepsilon}\psi\left(\frac{t}{\varepsilon}\right) \qquad \lambda^*(x) = \frac{1}{\delta}\lambda\left(\frac{x}{\delta}\right) \qquad \hat{\tilde{P}}^*(k,s) = \frac{1-\tilde{\psi}(\varepsilon s)}{s}\,\frac{1}{1-\tilde{\psi}(\varepsilon s)\,\hat{\lambda}(\delta k)}$$

Trapping pdf $0<\beta<1$

$$\psi(t) \sim t^{-(1+\beta)}$$

Jumps pdf $1<\alpha<2$

$$\lambda(x) \sim |x|^{-(1+\alpha)}$$

Scale-free transport

$$\langle x^2 \rangle \to \infty$$

$$\langle t \rangle \to \infty$$

Fluid limit $\quad \varepsilon \to 0 \quad \delta \to 0$

$$\tilde{\psi}(\varepsilon s) \approx 1 - c_1 (\varepsilon s)^\beta + \cdots$$

$$\hat{\lambda}(\delta k) \approx 1 - c_2 (\delta |k|)^\alpha + \cdots$$

$$\longrightarrow \quad s^\beta \hat{\tilde{P}} - s^{\beta-1} = -\chi |k|^\alpha \hat{\tilde{P}}$$

We still have to invert the Fourier-Laplace transforms!

Metzler, Klafter Phys. Reports **339**, 1 (2000)

Fractional diffusion equation

$$s^{\beta}\,\hat{\tilde{P}} - s^{\beta-1} = -\chi\,|k|^{\alpha}\,\hat{\tilde{P}}$$

For $\alpha = 2 \quad \beta = 1$

Laplace transform $\quad L\left[\partial_t P\right] = s\tilde{P} - \delta(x)$

Fourier transform $\quad F\left[\partial^2{}_x P\right] = -k^2\,\hat{P}$

Diffusion Equation

$$\partial_t P = \chi\,\partial_x^2 P$$

For $\alpha \neq 2 \quad \beta \neq 1$ define the operators:

Spatial fractional derivative $\quad D^{\alpha}_{|x|}P$

Temporal fractional derivative $\quad D^{\beta}_{t}P$

$$F\left[D^{\alpha}_{|x|}P\right] = -|k|^{\alpha}\,\hat{P}$$

$$L\left[D^{\beta}_{t}P\right] = s^{\beta}\,\tilde{P} - s^{\beta-1}\delta(x)$$

$$D^{\beta}_{t}P = \chi\,D^{\alpha}_{|x|}P$$

Fractional Diffusion Equation

What is a fractional derivative?

$$\frac{d^n f}{dx^n}$$

L'Hopital (1695):
"What if n=1/2?"

Leibniz (1695):
"This is an apparent paradox from which, one day, useful consequences will be drawn"

It is a usual practice to extend mathematical operations, originally defined for a set of objects, to a wider set of objects

$$\sqrt{r} \quad r \in R^+ \quad \Rightarrow \quad \sqrt{s} \quad s \in R$$

$$n! \quad n \in N^+ \quad \Rightarrow \quad \Gamma(s) \quad s \in R$$

$$\partial_x^n \phi \quad n \in N^+ \quad \Rightarrow \quad \partial_x^\alpha \phi \quad \alpha \in C$$

This mathematical "games" have, eventually, important physical applications

Fractional integral

Integer order integration

$$ {}_aD_x^{-n}\,\phi = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n\, \phi(x_n) $$

$$ {}_aD_x^{-n}\,\phi = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1}\,\phi(y)\,dy $$

Riemann-Liouville fractional integral

$$ {}_aD_x^{-\nu}\,\phi = \frac{1}{\Gamma(\nu)}\int_a^x (x-y)^{\nu-1}\,\phi(y)\,dy $$

Example: $\quad {}_0D_x^{-\nu}\, x^{\mu} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}\, x^{\mu+\nu}$

Fractional derivative

$$ {}_aD_x^{\mu}\,\phi = \frac{d^N}{dx^N}\left[{}_aD_x^{-\nu}\,\phi \right] $$

$\nu = N - \mu$

N = smallest integer $> \mu$

$$ {}_aD_x^{m}\,\phi = \frac{d^N}{dx^N}\left[{}_aD_x^{-(N-m)}\,\phi \right] = \frac{d^m}{dx^m}\,\phi $$

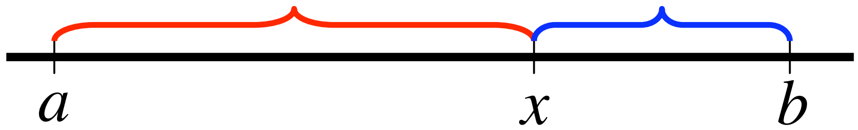
Riemann-Liouville fractional derivative

$$ {}_aD_x^{\alpha}\,\phi = \frac{1}{\Gamma(2-\alpha)}\,\partial_x^2 \int_a^x \frac{\phi(y)}{(x-y)^{\alpha-1}}\,dy \qquad 1 < \alpha \le 2 $$

Examples: $ {}_0D_x^{\mu}\,x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\,x^{\lambda-\mu} $ $\qquad$ $ {}_{-\infty}D_x^{\mu}\,e^{ikx} = (ik)^{\mu}\,e^{ikx} $

Fractional derivatives: General definition

•In real space, fractional derivative are non-local, integro-differential operators



$$n-1<\alpha<n$$

<u>Left derivative</u>

$$ {}_aD_x^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_a^x \frac{f(u)}{(x-u)^{\alpha-n+1}} du $$

<u>Right derivative</u>

$$ {}_xD_b^\alpha f = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_x^b \frac{f(u)}{(u-x)^{\alpha-n+1}} du $$

<u>General spatial derivative</u>

$$ D_x^\alpha f = \left[l\ {}_aD_x^\alpha + r\ {}_xD_b^\alpha \right] f $$

<u>Time derivative</u>

$$ D_t^\beta f = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial_\tau f(\tau)}{(t-\tau)^\beta} d\tau $$

•Fractional derivatives behave in some sense like regular derivatives, and many results (not all!) easily generalize to fractional order.

Podlubny, *Fractional differential equations* (Academic Press, San Diego, 1999).

Fractional diffusion equation: flux conserving form

$$\partial_t T = -\partial_x q + S$$

•Fractional diffusion assumes a non-local flux-gradient relation of the form

$$q_{nl}(x,t) = -\chi \quad \partial_x \left[l \int_a^x \frac{T(y,t)}{(x-y)^{\alpha-1}} dy + r \int_x^b \frac{T(y,t)}{(y-x)^{\alpha-1}} dy \right]$$

$1 < \alpha < 2$

$\beta = 1$

•In Fourier space:

$$\hat{q}_{nl} = -\chi \left[l(-ik)^{\alpha-1} - r(ik)^{\alpha-1} \right] \hat{T}(k)$$

•The scaling $\hat{q}_{nl} \sim (ik)^{\alpha-1}$ motivates the term name fractional diffusion

•In contrast, in standard diffusive paradigm

$$q_d = -\chi_d \, \partial_x T \qquad \partial_t T = \chi \, \partial^2{}_x T + S \qquad \hat{q}_d \sim ik \, \hat{T}(k)$$

•This local description assumes a well-defined transport scale and that widely separated regions do not interact.

Fractional diffusion equation: fundamental solution

$$\partial_t^\beta \phi = \chi\, \partial_{|x|}^\alpha \phi \qquad \phi(x, t=0) = \delta(x)$$

Fourier Laplace transforms

$$F\left[\partial_{|x|}^\alpha \phi\right](k) = -\,|k|^\alpha\, \phi(k) \qquad L\left[\partial_t^\beta \phi\right](s) = s^\beta\, \phi(s) - s^{\beta-1}\phi(0)$$

$$\phi\,(k,s) = \frac{s^{\beta-1}}{s^\beta + \chi |k|^\alpha} \qquad E_\beta(z) = \sum_n \frac{z^n}{\Gamma(\beta\, n+1)}$$

Mittag-Leffler function

$$\phi\,(x,\,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx}\, E_\beta\left(-\chi\,|k|^\alpha\, t^\beta\right) dk$$

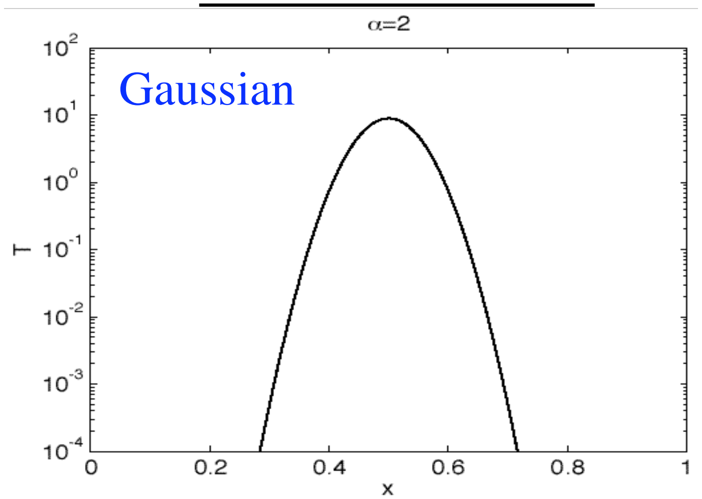
$$\beta = 1 \;\rightarrow\; E_\beta(z) = \mathrm{e}^z \begin{cases} \alpha \neq 2 & \phi\,(x,\,t) = \alpha - \text{Stable Levy distribution} \\ \alpha = 2 & \phi\,(x,\,t) = \text{Gaussian} \end{cases}$$

The fundamental solutions of the fractional diffusion equation are the Levy stable distributions

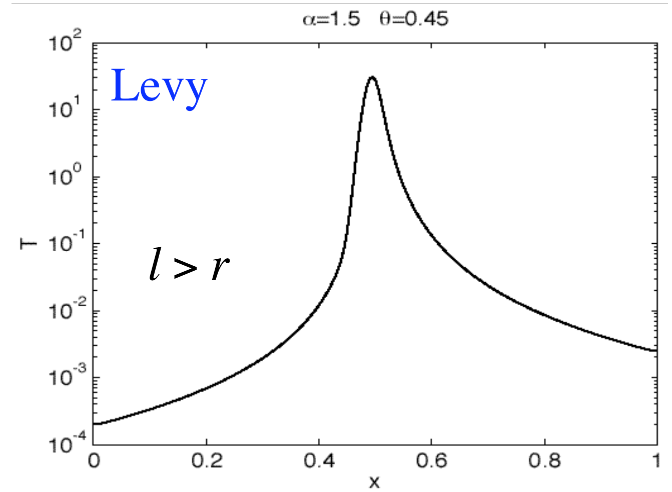
Standard diffusion

$\alpha = 2$

$\beta = 1$



Asymmetric fractional diffusion



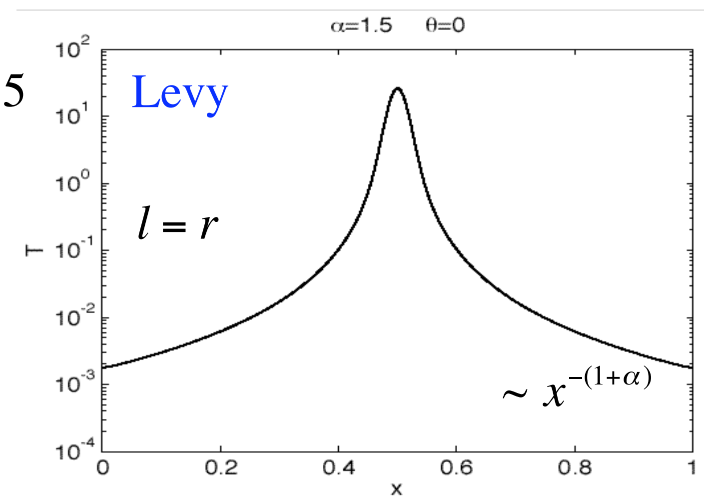
$\alpha = 1.5$

$\beta = 1$

Symmetric fractional diffusion

$\alpha = 1.5$

$\beta = 1$



$$P\left(x,\lambda t\right) = \frac{1}{\lambda^{\beta/\alpha}} P\left(\frac{x}{\lambda^{\beta/\alpha}}, t\right)$$

Self-similar scaling

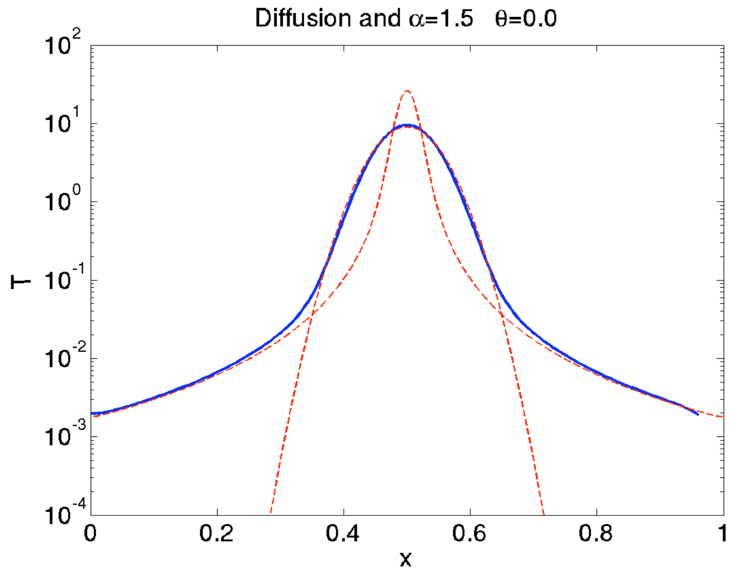
$$\left\langle x^n \right\rangle \sim t^{n\beta/\alpha}$$

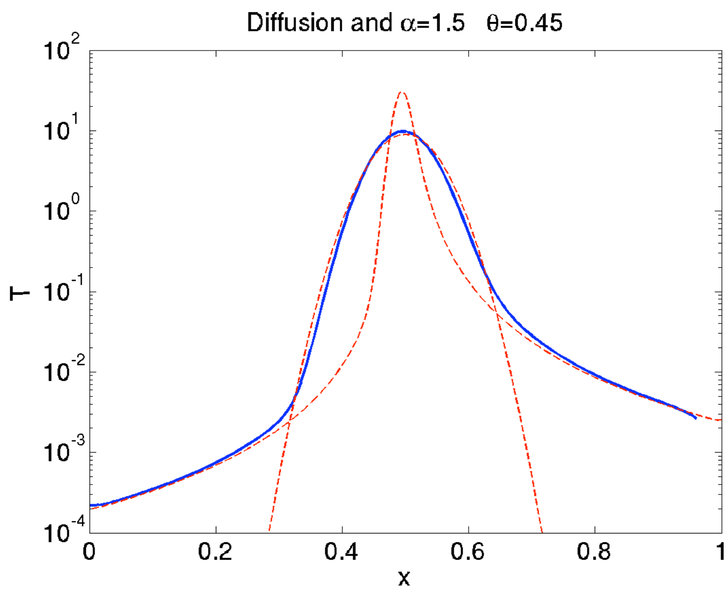
Non-diffusive scaling

$$P \sim x^{-(1+\alpha)}$$

Algebraic decay

Interplay of regular and fractional diffusion





$$\partial_t T = \chi_f \left[l \, {}_aD_x^\alpha + r \, {}_xD_b^\alpha \right] T + \chi_d \, \partial_x^2 T$$

$$T(x, t=0) = \delta(x)$$

$$T(x=0,t) \sim \begin{cases} (\chi_d \, t)^{-1/2} & \text{for} \quad t \sim 0 \\ (\chi_f \, t)^{-1/\alpha} & \text{for} \quad t >> 1 \end{cases}$$

Algebraic decay and nonlocal transport

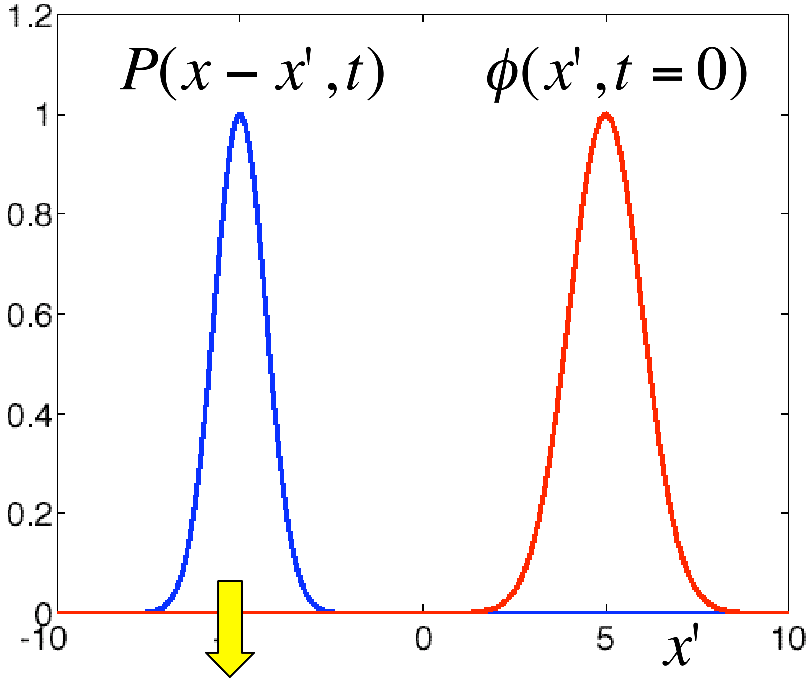
$$\phi(x,t) = \int_{-\infty}^{\infty} dx' \; P(x - x' ; t)\, \phi(x', t = 0)$$

$$\lim_{x\to\infty} P \sim e^{-x^2}$$

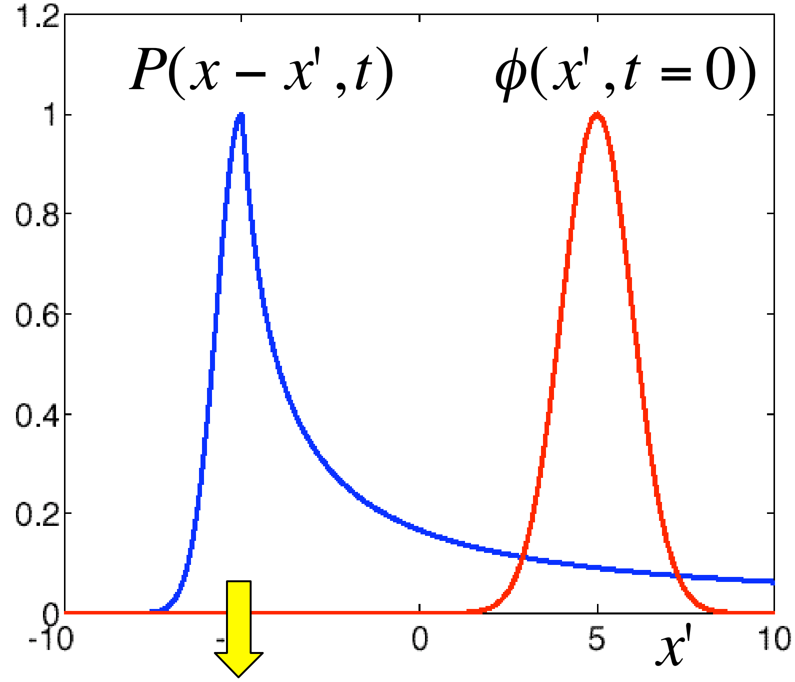
local transport

nonlocal transport

$$\lim_{x\to\infty} P \sim x^{-(1+\alpha)}$$



$$\phi(x,t) \approx 0$$



$$\phi(x,t) \neq 0$$

Due to the long-range, algebraic decay of the propagator, fractional diffusion leads to nonlocal transport

Applications 1: Transport in plasma turbulence

Pressure gradient-driven turbulence model

$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\nabla_\perp^2\tilde{\Phi} = \frac{B_0}{m_i n_0 r_c}\frac{1}{r}\frac{\partial\tilde{p}}{\partial\theta} - \frac{1}{\eta m_i n_0 R_0}\nabla_\parallel^2\tilde{\Phi} + \mu\nabla_\parallel^4\tilde{\Phi}$$

$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\tilde{p} = \frac{\partial\langle p\rangle}{\partial r}\frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial\theta} + \chi_\perp\nabla_\perp^2\tilde{p} + \chi_\parallel\nabla_\parallel^2\tilde{p}$$

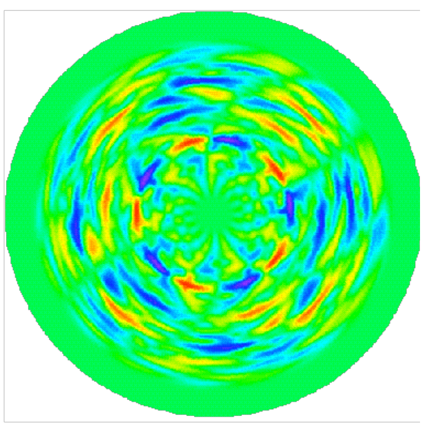
$$\frac{\partial\langle p\rangle}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}r\left\langle\tilde{V}_r\,\tilde{p}\right\rangle = S_0 + D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\langle p\rangle}{\partial r}\right)$$

Tracers evolution

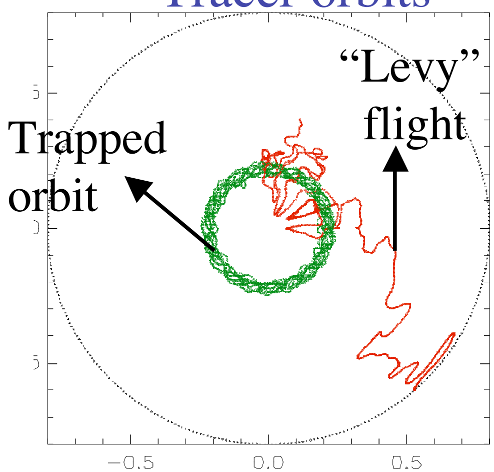
$$\frac{d\vec{r}}{dt} = \frac{1}{B^2}\nabla\tilde{\Phi}\times\vec{B}$$

Super-diffusive scaling

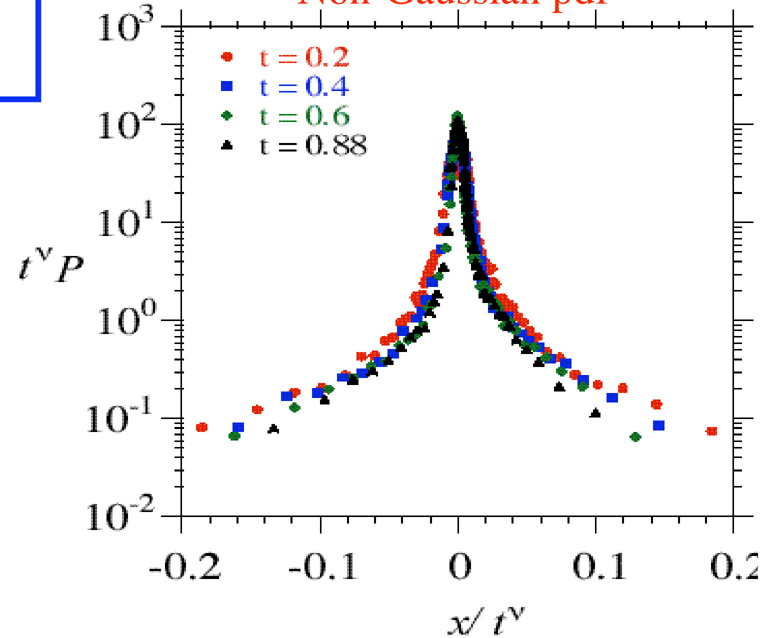
$$\left\langle \delta r^n \right\rangle \sim t^{2n/3}$$



Tracer orbits



Non-Gaussian pdf



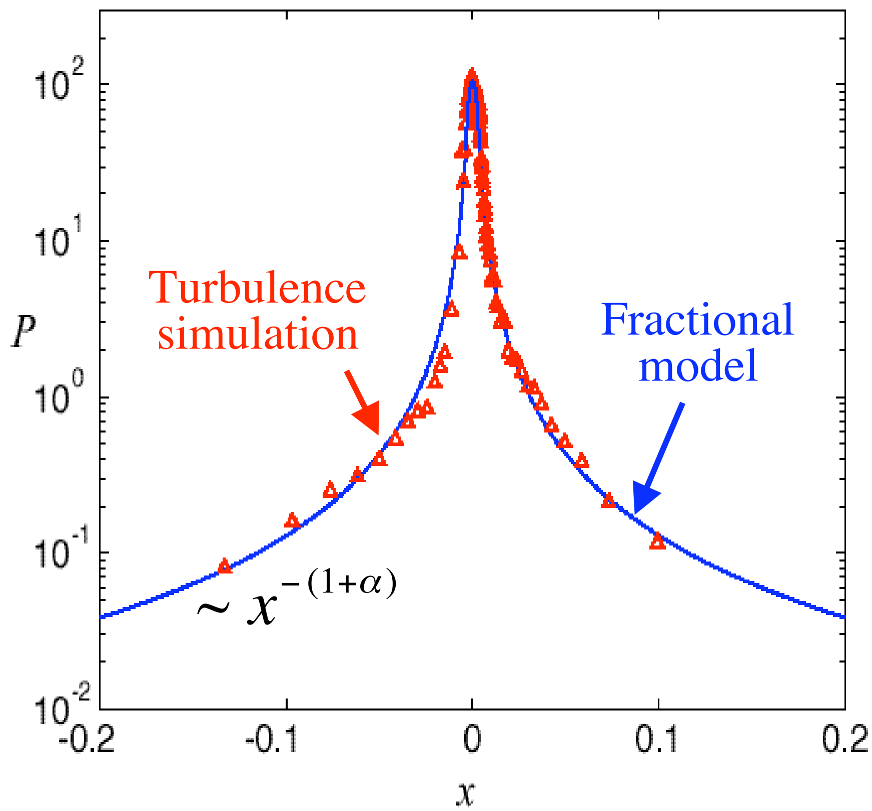
D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

Comparison between fractional model and turbulent transport data

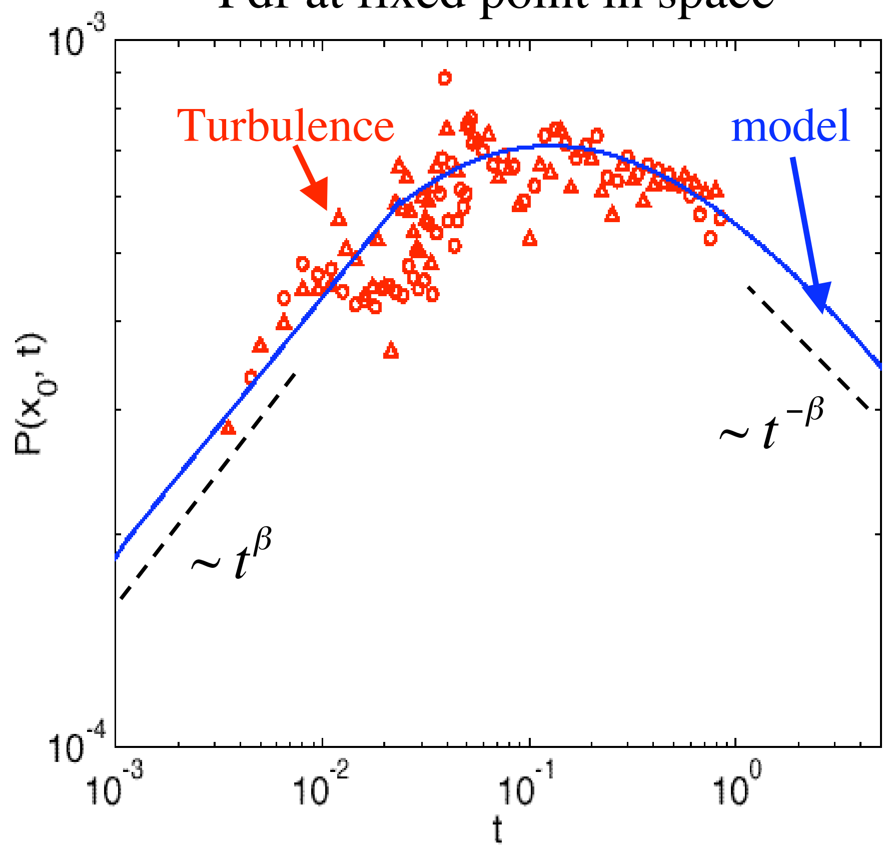
$$\alpha = 3/4 \qquad \beta = 1/2$$

$$\langle x^2 \rangle \sim t^{2\beta/\alpha} \sim t^{4/3}$$

Levy distribution at fixed time



Pdf at fixed point in space



D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

Effective transport operators for turbulent transport

Individual tracers move following the turbulent velocity field

$$\frac{d\vec{r}}{dt} = \tilde{V} = \frac{1}{B^2}\nabla\tilde{\Phi} \times \vec{B}$$

The distribution of tracers P evolves according the passive scalar equation

$$\frac{\partial P}{\partial t} = -\tilde{V} \cdot \nabla P$$

The proposed model encapsulates the Spatio-temporal complexity of the turbulence using fractional operators in space and time

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[q_\ell + q_r \right]$$

Fractional derivative operators are useful tools to construct effective transport operators when Gaussian closures do not work

$$\left[\partial_t + \tilde{V} \cdot \nabla\right] \Leftrightarrow \left[{}_0D_t^\beta - \chi\left({}_aD_x^\alpha + {}_xD_b^\alpha\right)\right]$$

Fractional approach

$$\left(\partial_t + \tilde{V} \cdot \nabla\right) \Leftrightarrow \left(\partial_t - \chi\, \partial_x^2\right)$$

Gaussian approach

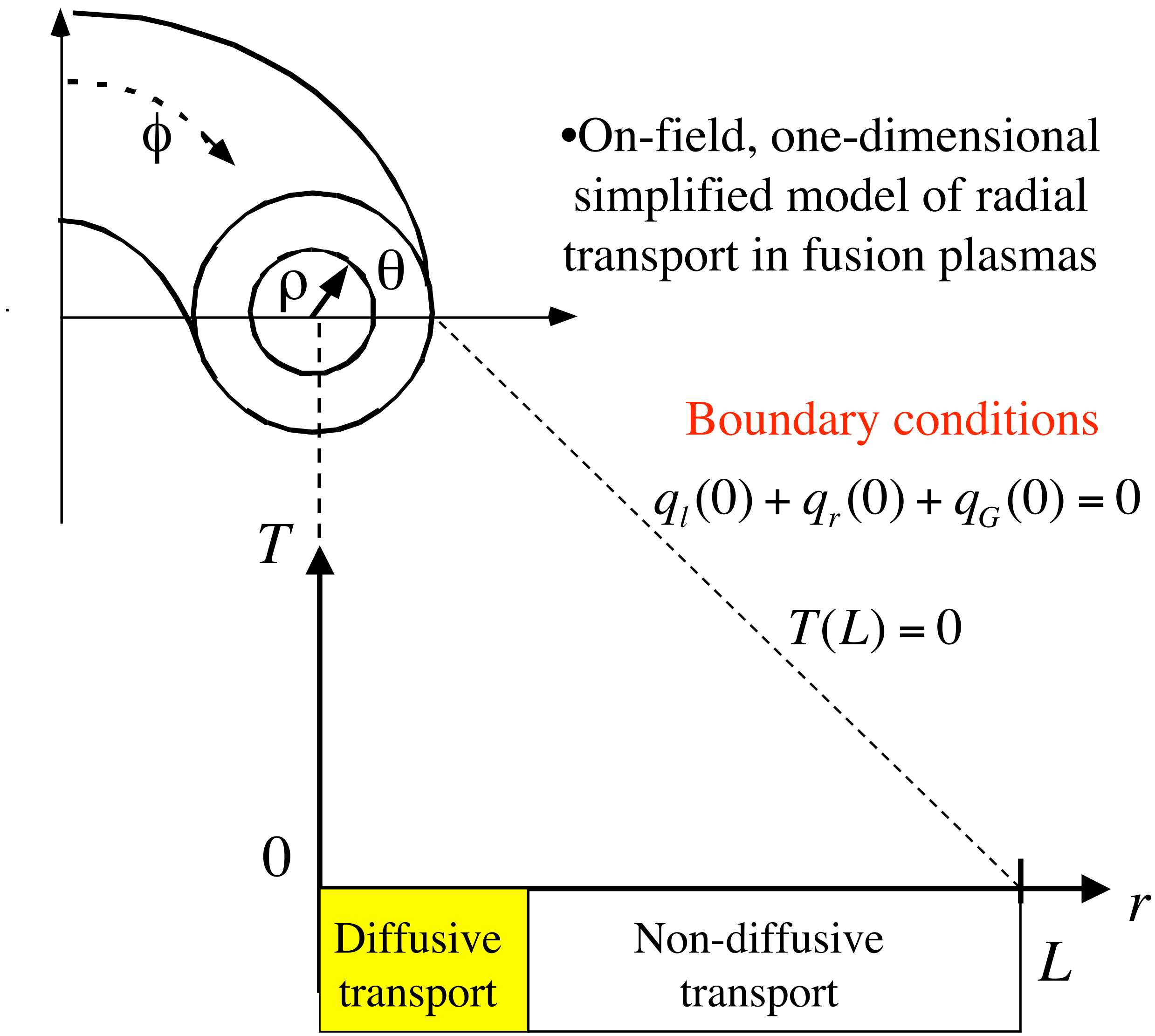
Finite size domain transport problem

•Fractional diffusion in unbounded domains with constant diffusivities is well understood

•However, finite size domains with variable diffusivities and boundary conditions introduce complications

•Understanding this problem is critical for applications of fractional diffusion

•On-field, one-dimensional simplified model of radial transport in fusion plasmas



Finite-size domains:
Regularization of fractional derivatives

•Finite size model are non-trivial because the truncate (finite *a* and *b*) RL fractional derivatives are in general singular at the boundaries

$1 < \alpha < 2$

$$
{}_aD_x^\alpha\,\phi = \underbrace{\frac{\phi(a)}{\Gamma(1-\alpha)}\frac{1}{(x-a)^\alpha} + \frac{\phi'(a)}{\Gamma(2-\alpha)}\frac{1}{(x-a)^{\alpha-1}}}_{\text{singular terms}} + \underbrace{\sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)\left(x-a\right)^{k+2-\alpha}}{\Gamma\left(k+3-\alpha\right)}}_{\text{regular terms}}
$$

These problems can be resolved by defining the fractional operators in the Caputo sense.

$$
{}_aD_x^\alpha\left[\phi(x)-\phi(a)-\phi^{(1)}(a)\right] = \sum_{k=0}^{\infty}\frac{\phi^{(k+2)}(a)\left(x-a\right)^{k+2-\alpha}}{\Gamma\left(k+3-\alpha\right)}
$$

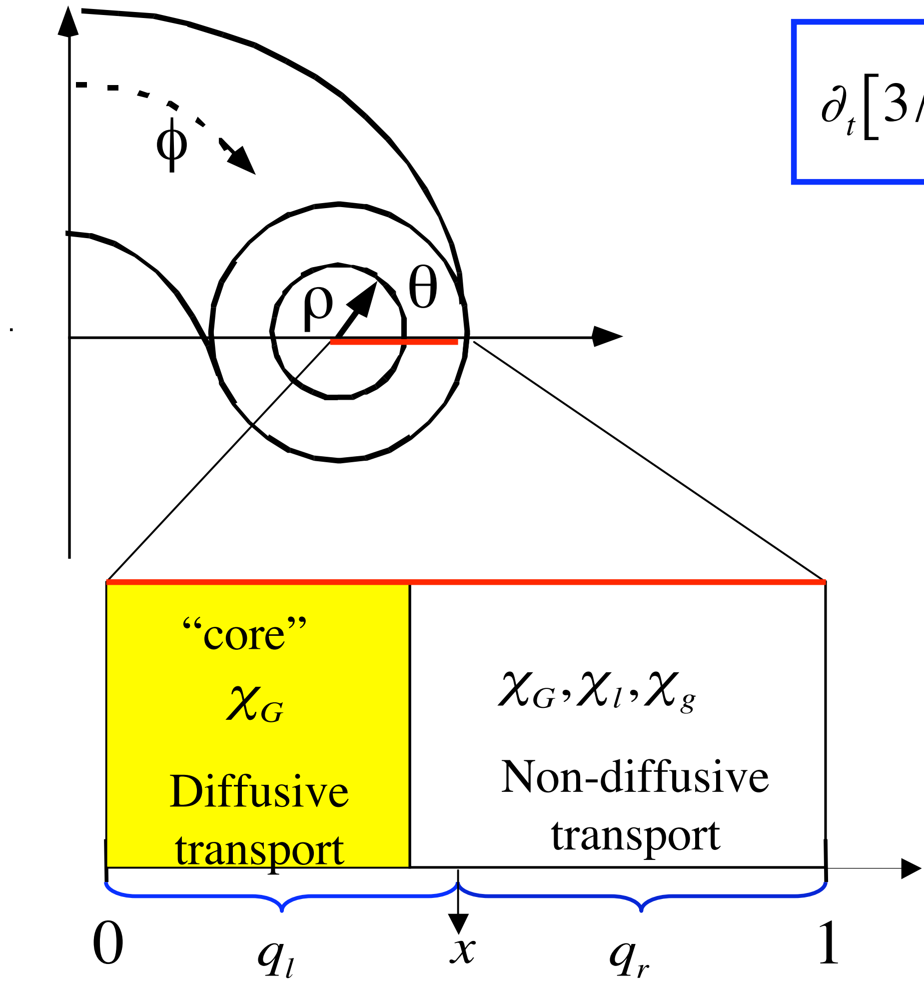
We define the regularized left derivative as

$$
{}_0^CD_x^\alpha\phi = {}_aD_x^\alpha\left[\phi(x)-\phi(a)-\phi^{(1)}(a)\right]
$$

$$
\boxed{{}_a^CD_x^\alpha\,\phi = \frac{1}{\Gamma(2-\alpha)}\int_a^x\frac{\phi''(u)}{\left(x-u\right)^{\alpha-1}}\,du}
$$

A similar calculation gives the regularized right derivative as

Finite-size domain fractional diffusion model



$$\partial_t\left[3/2 n_e T_e\right] = -\,\partial_x\left[\, q_\ell + q_r + q_d \right] + S(x,t)$$

Local flux $\quad q_d = -n_e\,\chi_d\,\partial_x\,T$

Nonlocal fluxes

$$\begin{cases} q_\ell = -\,l\,n_e\,\chi_{nl}\;{}_0D_x^{\alpha-1}\,T \\ q_r = r n_e \chi_{nl}\;{}_xD_1^{\alpha-1}\,T \end{cases}$$

Regularized fractional derivatives

$${}_0D_x^{\alpha-1}\,T = \frac{1}{\Gamma(3-\alpha)}\int_0^x (x-y)^{2-\alpha}\,T''\,dy$$

$${}_xD_1^{\alpha-1}\,T = \frac{1}{\Gamma(3-\alpha)}\int_x^1 (y-x)^{2-\alpha}\,T''\,dy$$

Boundary conditions

$$\left[q_l + q_r + q_G\right](0) = 0 \qquad T(1) = 0$$

D. del-Castillo-Negrete, Phys. Plasmas **13,** 082308 (2006)

Numerical method

- Finite difference definition of fractional operators (Grunwald-Letnikov)

$$ {}_aD_x^\alpha \phi = \lim_{h\to 0} \frac{{}^{-}\Delta_h^\alpha \phi}{h^\alpha} \qquad {}^{-}\Delta_h^\alpha \phi = \sum_{j=0}^{K_\mp} w_j^{(\alpha)} \phi(x - jh) $$

- Triangular matrix, finite difference representation very useful for the numerical solution of fractional transport models:

$$ \mathbf{D}^\alpha = \begin{pmatrix} w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & \cdots & 0 \\ w_2^{(\alpha)} & w_1^{(\alpha)} & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots 0 & \cdots \\ w_{N-1}^{(\alpha)} & w_{N-2}^{(\alpha)} & & \cdots & 0 \end{pmatrix} $$

- Stable, efficient and accurate finite difference codes for the solution of non-local transport equations have been developed.

Lynch, et.al. J. Comp. Phys. **192**, 2 406 (2003)

D. del-Castillo-Negrete, Phys. Plasmas **13,** 082308 (2006)

Applications 2:
Anomalous confinement time scaling

$$\tau = \frac{\int_0^L T\,dx}{\int_0^L P\,dx}$$

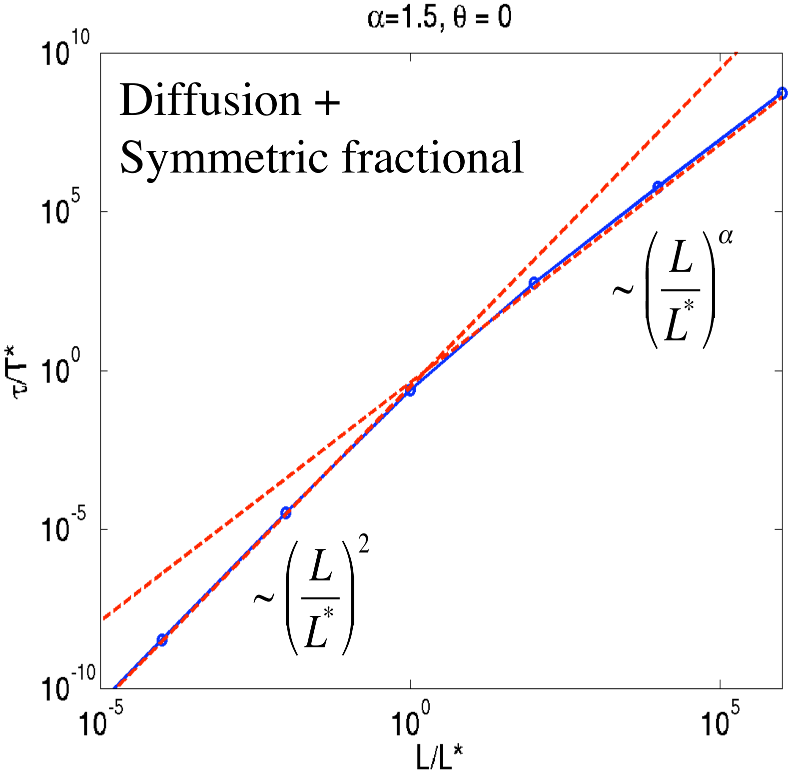
$$L^* = \left(\frac{\chi_d}{\chi_a}\right)^{\frac{1}{2-\alpha}} \qquad T^* = \left(\frac{\chi_d^\alpha}{\chi_a^2}\right)^{\frac{1}{2-\alpha}}$$

τ = Confinement time

L = Domain size

χ_d = Standard diffusivity

χ_a = Anomalous diffusivity



$L << L^*$	Diffusive scaling	$\tau \sim \chi_d^{-1}\, L^2$
$L >> L^*$	Non-diffusive scaling	$\tau \sim \chi_f^{-1}\, L^\alpha$

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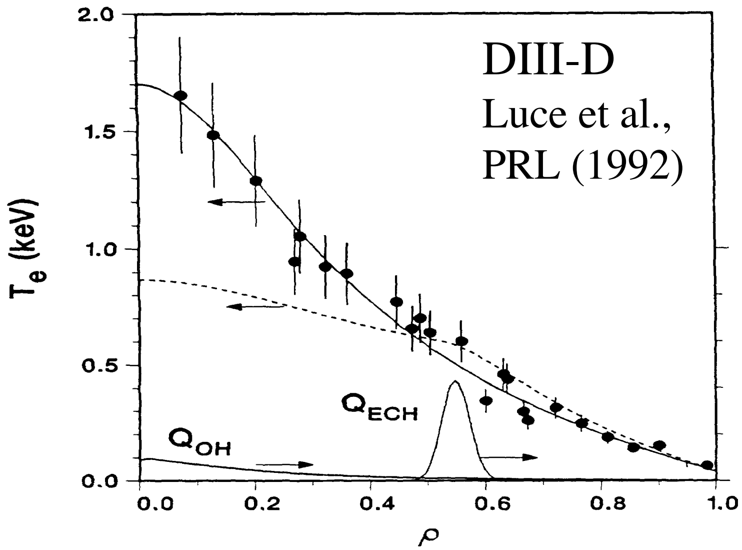
Applications 3: Profile peaking

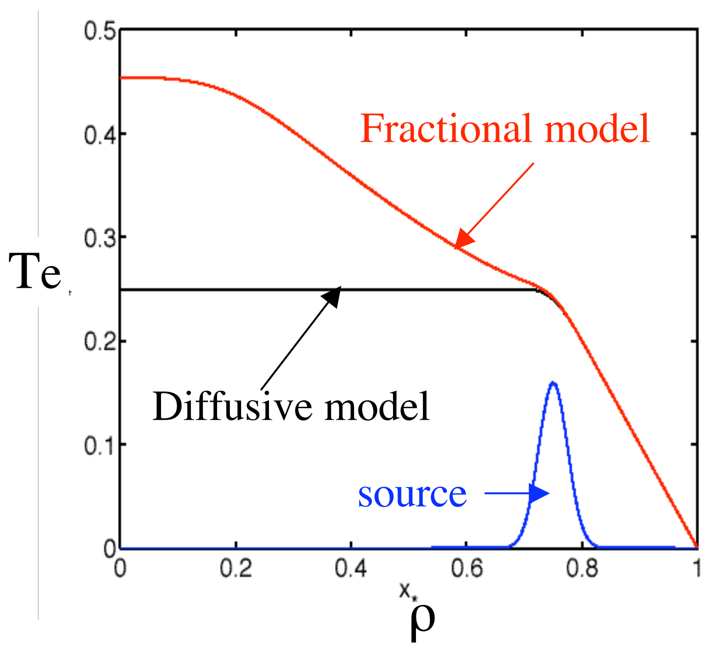


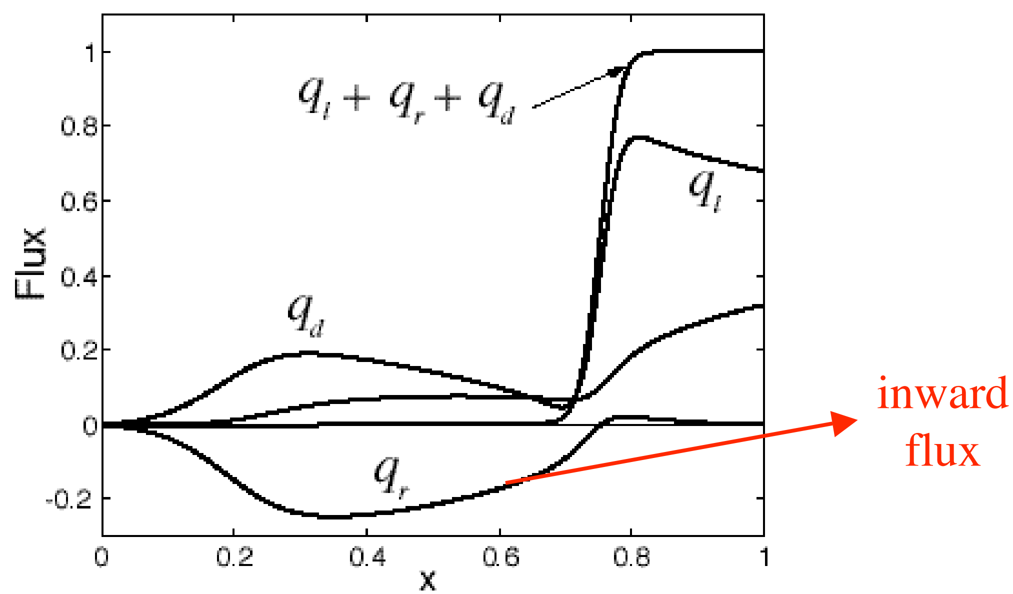
$$\partial_t T_e = -\partial_x \left[q_\ell + q_r + q_d \right] + S(x,t)$$

$$\begin{cases} q_\ell \sim \partial_x \int_0^x T_e \cdots & \text{Left flux} \\ q_r \sim \partial_x \int_x^1 T_e \cdots & \text{Right flux} \end{cases}$$

Local flux

$$q_d \sim \partial_x T_e \cdots$$





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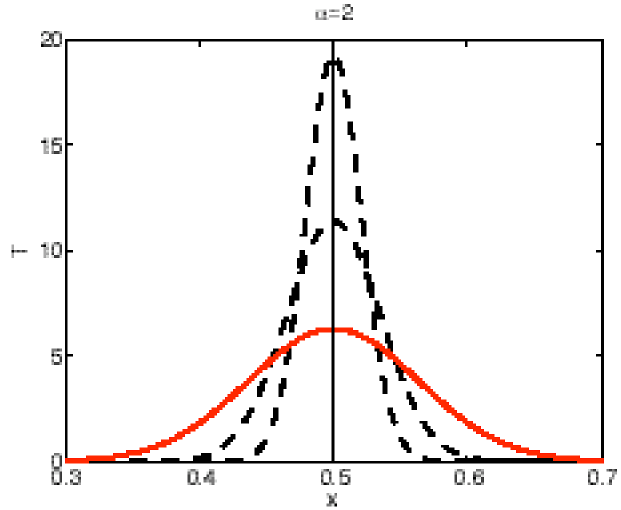
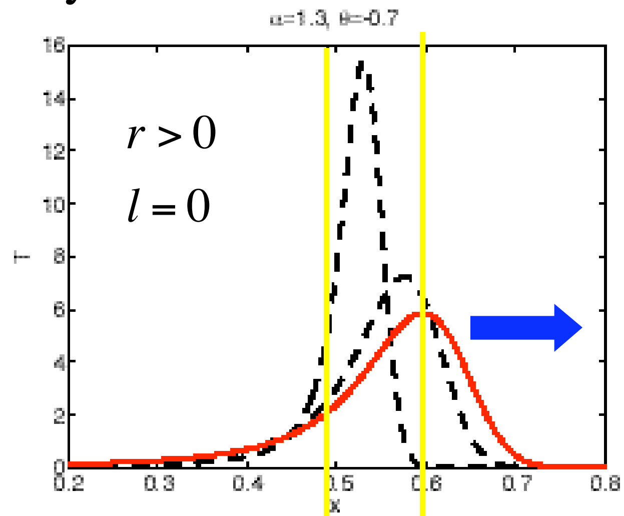
Applications 4: Pinch effects and up-hill transport

Standard diffusion



Asymmetric fractional diffusion



$$V \approx \chi_f^{1/\alpha}\, \theta \left(\frac{\alpha+1}{2\alpha}\right) \alpha^{1/\alpha} \left|\tan\left(\frac{\alpha\pi}{2}\right)\right|$$

$$x_m = V\, t^{\beta/\alpha}$$

$$q_G = -\chi_G\, \partial_x T$$

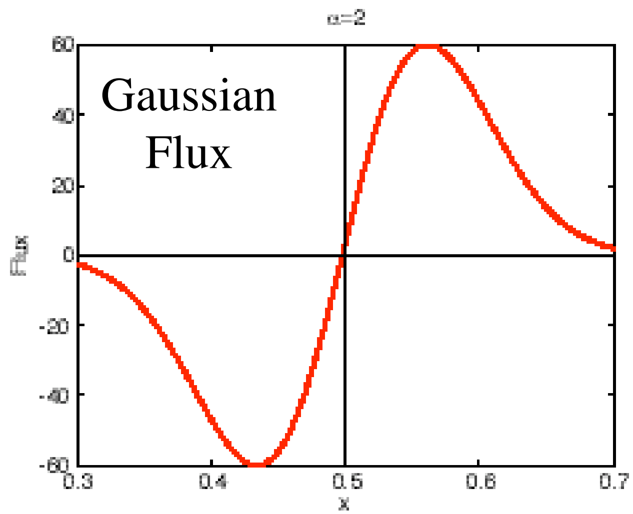
$$q_r = r\, {}_xD_1^{\alpha-1}\, T$$

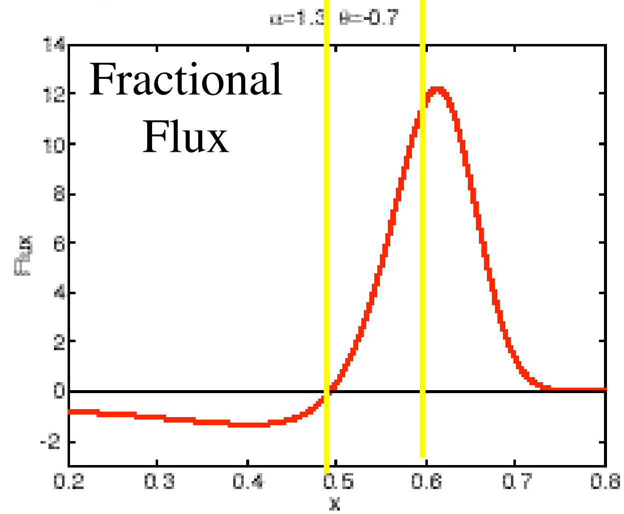
Up-hill
Non-Fickian
"anti diffusive zone"

↓

q and ∇T

have the **same** sign





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Applications 5. Perturbative transport: cold pulses and power modulation

•Perturbative transport experiments follow the transient response of the plasma to externally applied small perturbations, e.g. plasma edge cooling and ICRH heating power modulation.

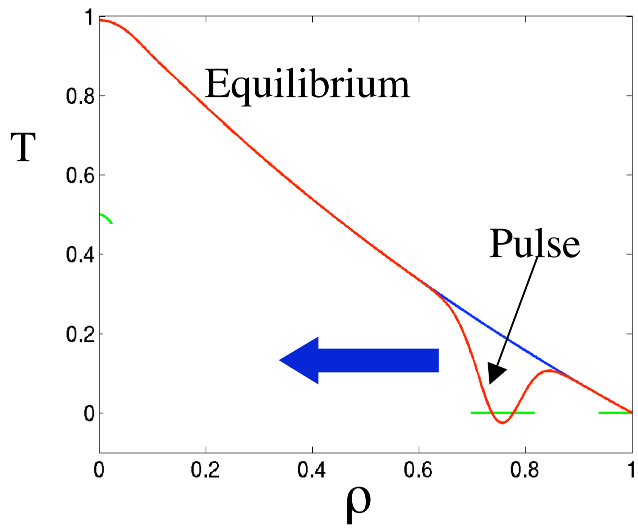
•They provide valuable time dependent transport information useful for validating and testing transport models

•One puzzling piece of evidence is that, while heat waves from ICRH power modulation can be adequately explained by local transport models, the speed of cold pulses is significantly underestimated.

•Local transport models not able to reproduce these results

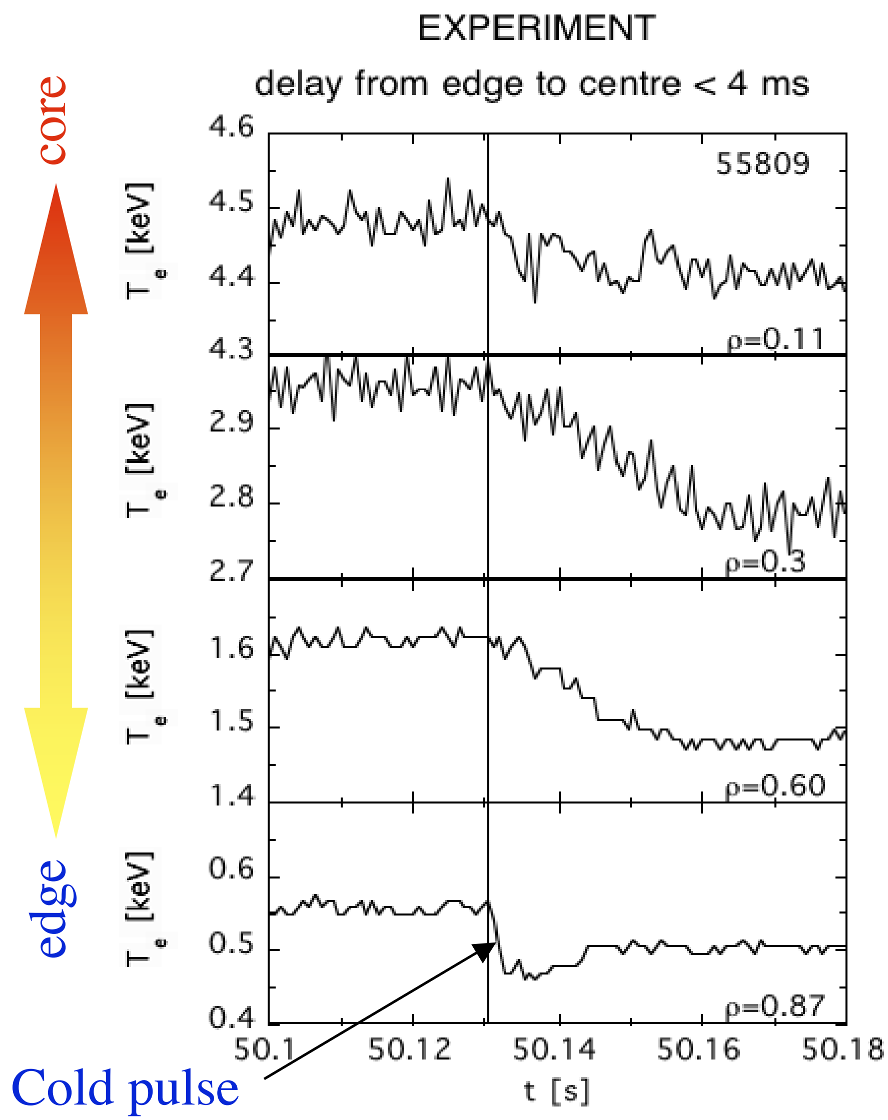
•Non-local model based on the use of fractional diffusion operators are capable of reproducing the experimental phenomenology.

Cold pulse propagation: experimental results



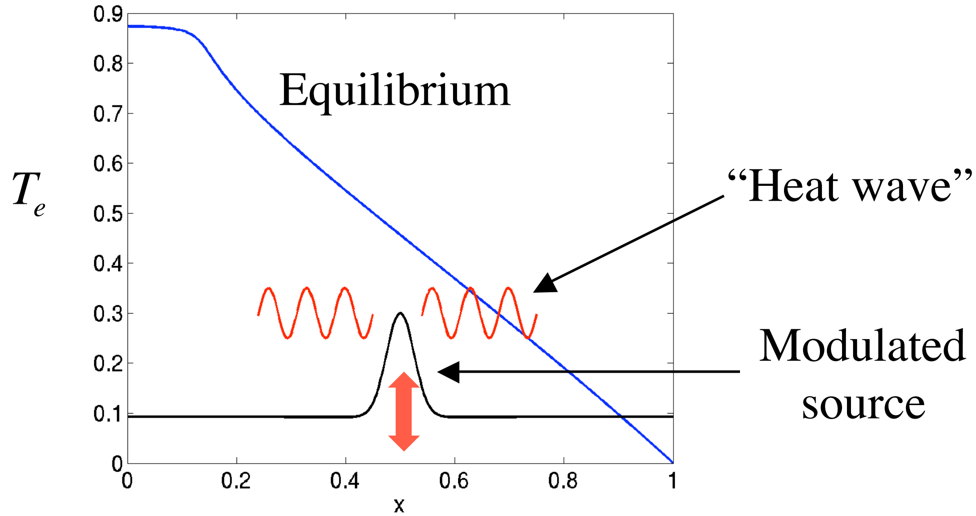
•Several observations at JET, and other Machines, indicate that cold pulses from the edge reach the plasma center very fast (~4 ms).

•Is the fast core response consistent with non-linear but local transport or does it require non-local transport?

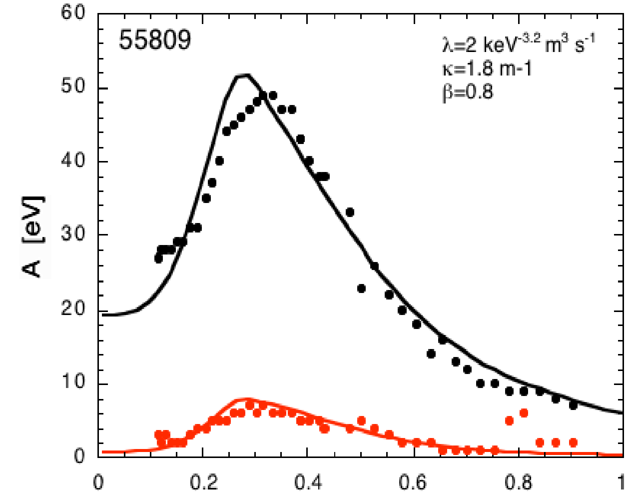


Mantica et al, 19th IAEA Conference EX/P1-04 (2002)

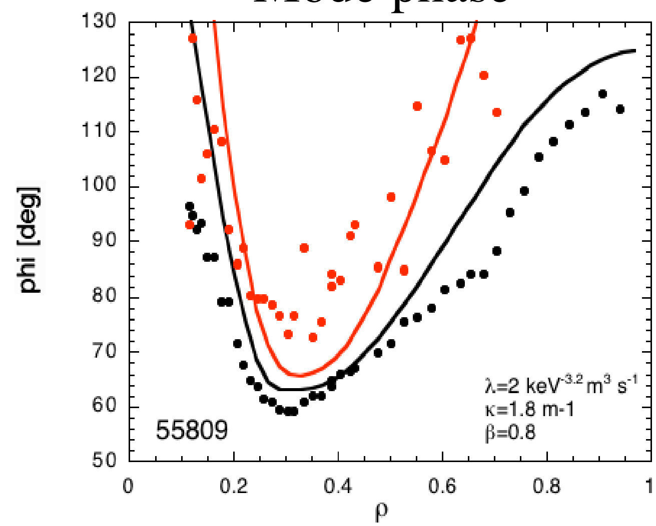
Power modulation: experimental results



Mode amplitude



Mode phase



$$T_e(\rho,t) = \sum_j A_j(\rho) \exp\left[i\phi_j(\rho) + i\omega_j t\right]$$

ρ>0.3: waves and pulses propagate fast

ρ<0.3 heat wave slows down and is damped

BUT cold pulse still travels fast!

This asymmetry seems incompatible with local transport models

Mantica et al, 19th IAEA Conference EX/P1-04 (2002)

Fractional diffusion model

$$\partial_x\left[3/2 n_e T_e\right] = -\partial_x\left[q_\ell + q_r + q_d\right] + S(x,t)$$

Source

$$S = P_0(x) + P_m(x,t)$$

$$P_0 = P_{OH} + P_{eNBI} + \frac{1}{2}\left[P_{RFmc} + P_{RFfw}\right]$$

$$P_m = \frac{A(t)}{2}\left[P_{RFmc} + P_{RFfw}\right]$$

Local flux $q_d = -n_e \chi_d \, \partial_x T$

Nonlocal fluxes

$$\begin{cases} q_\ell = -l\, n_e \chi_{nl} \; {}_0D_x^{\alpha-1} T \\ q_r = r n_e \chi_{nl} \; {}_xD_1^{\alpha-1} T \end{cases}$$

Density $n_e = 2.6 \times 10^{19}\,\mathrm{part}/m^3$

Model parameters

$$\alpha = 1.25$$

$$\chi_{nl} = 2\,\mathrm{m}^\alpha/\mathrm{sec} \quad x > 0.1$$

$$\chi_d = (0.75 + 6x)\,\mathrm{m}^2/\mathrm{sec}$$

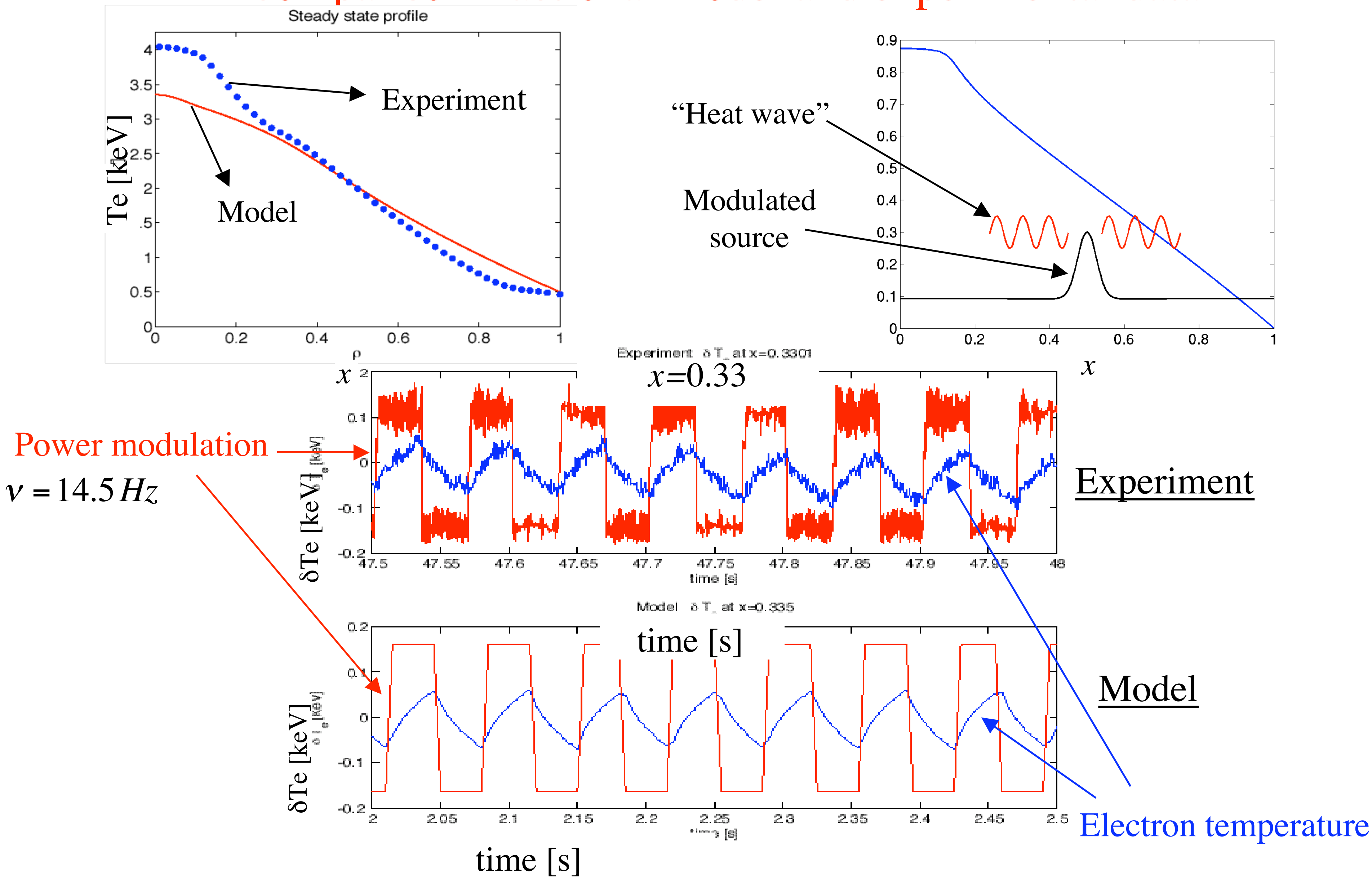
Regularized fractional derivatives

$${}_0D_x^{\alpha-1} T = \frac{1}{\Gamma(3-\alpha)} \int_0^x (x-y)^{2-\alpha} T'' \, dy$$

$${}_xD_1^{\alpha-1} T = \frac{1}{\Gamma(3-\alpha)} \int_x^1 (y-x)^{2-\alpha} T'' \, dy$$

Power Modulation:

comparison fractional model and experimental data



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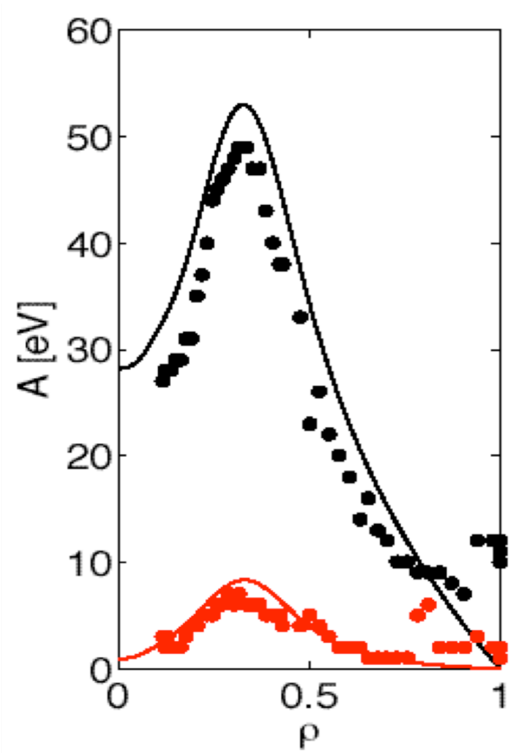
Power Modulation

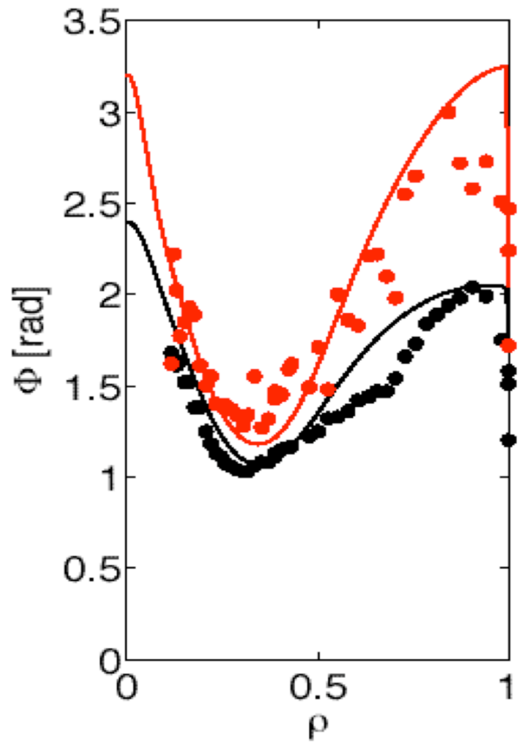
comparison fractional model with experiment

$$T_e(\rho,t) = \sum_j A_j(\rho)\exp\left[i\phi_j(\rho) + i\omega_j t\right]$$

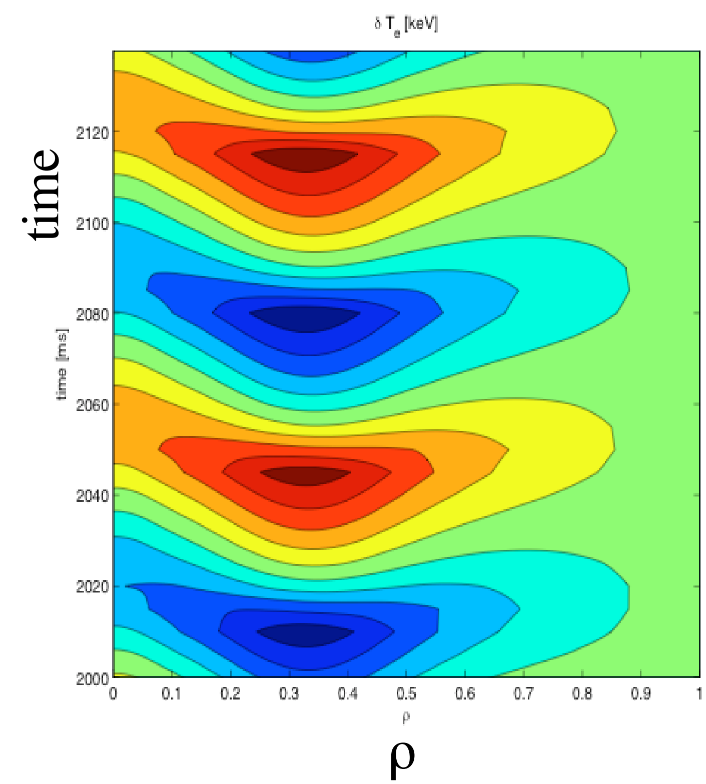
Mode amplitude



Mode phase



Space-time contour plot



black:1st harmonic
red: 3rd harmonic

dots: experiment
solid line: model

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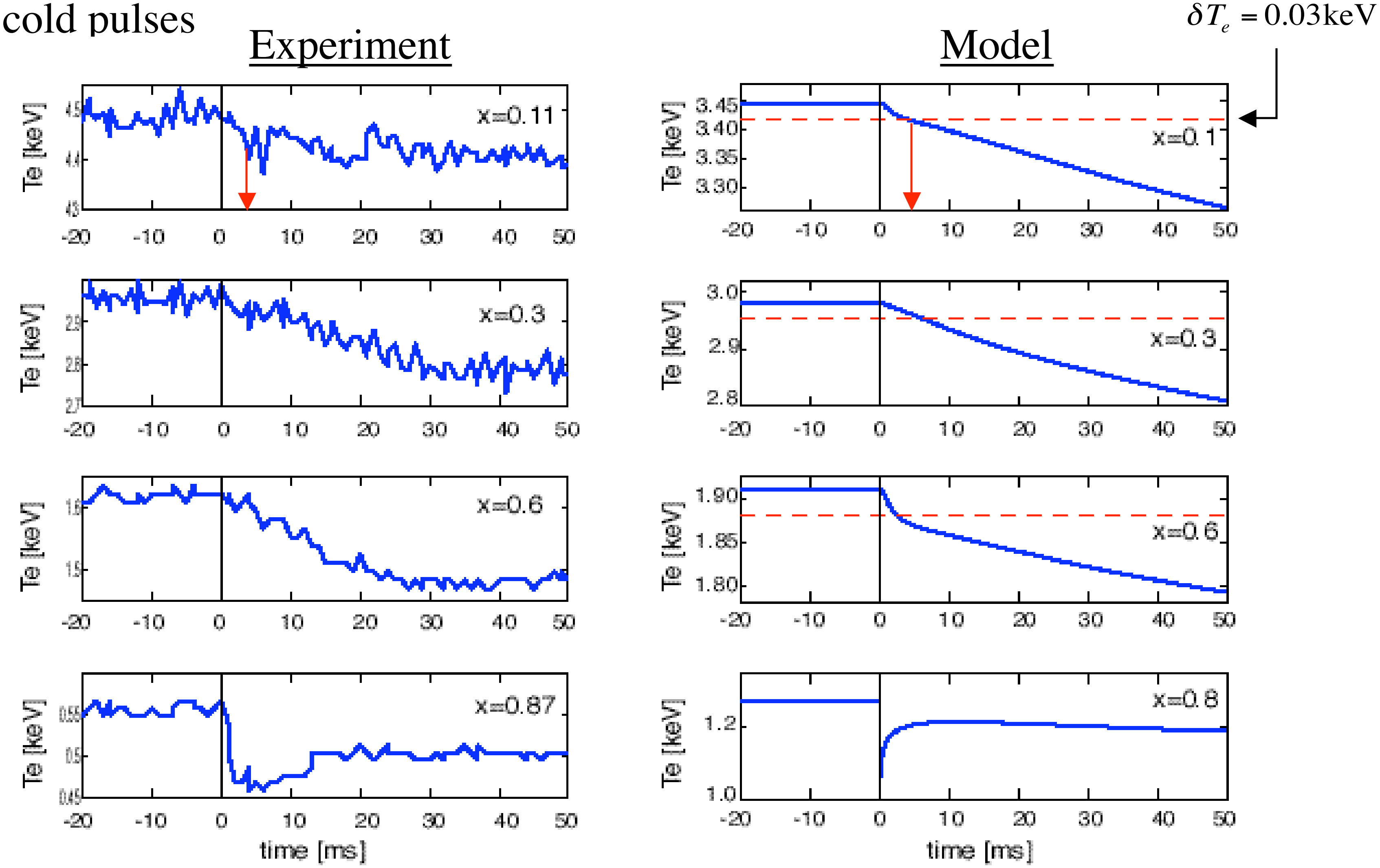
Cold Pulse: comparison fractional model with experiment

•Consistent with the experiment, the fractional model gives a delay of the order of 4ms for cold pulses



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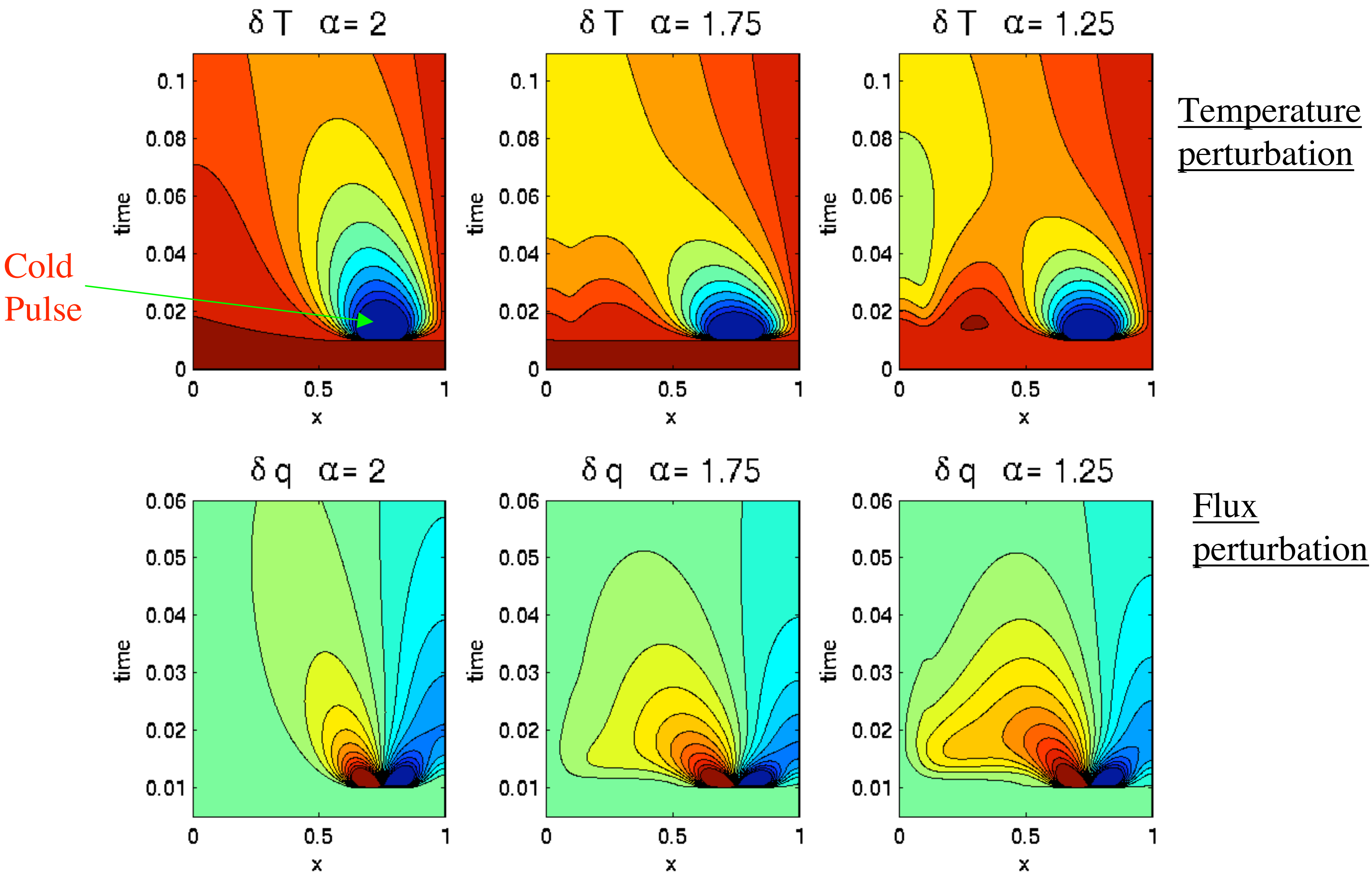
Dependence on non-locality parameter α

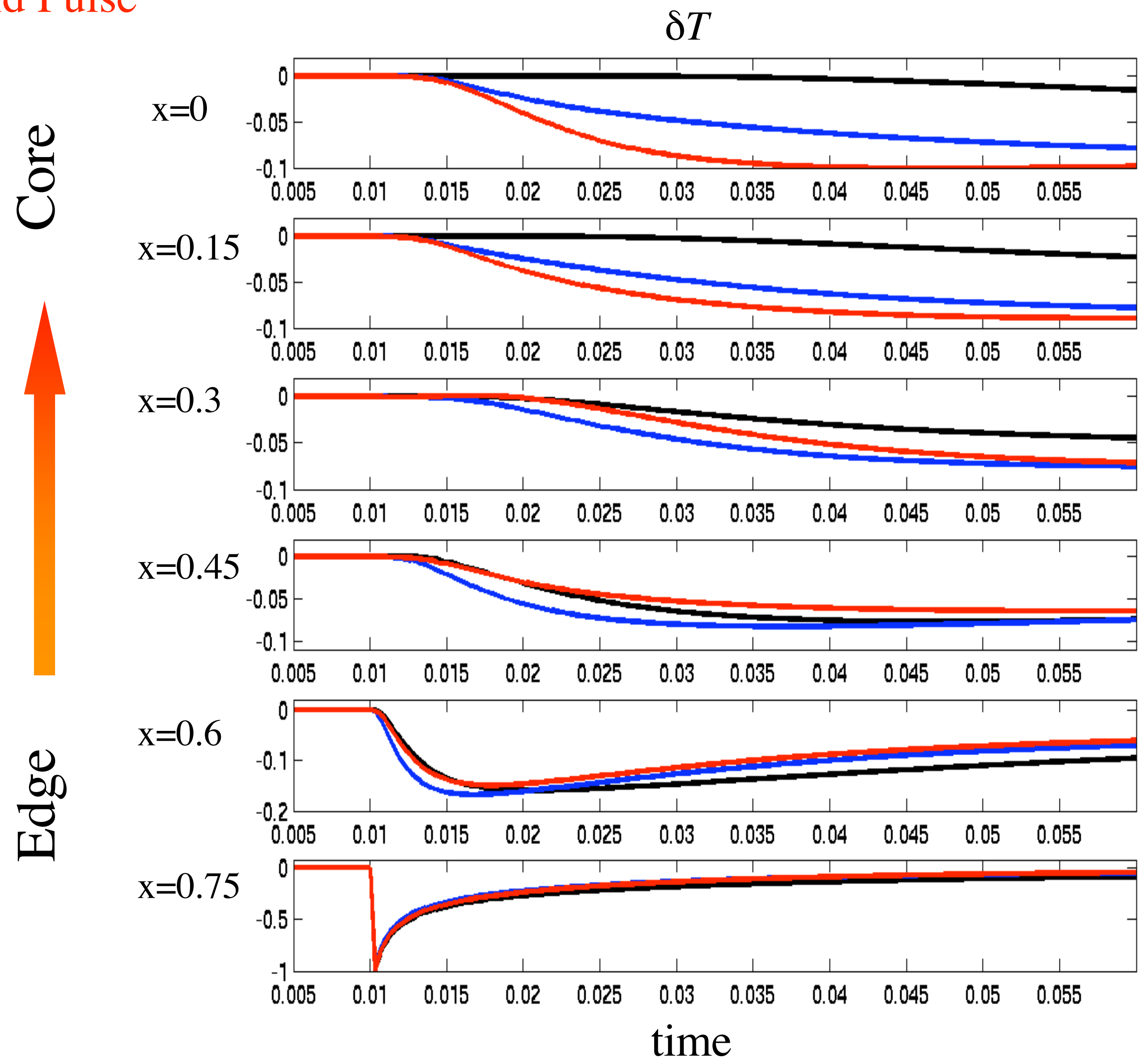
(symmetric case θ=0, stiffness χ_s =1)



Dependence on non-locality parameter α

(symmetric case θ=0, stiffness χ_s =1)

Cold Pulse



Red α=1.25

V=9.6

Blue α=1.75

V=6.3

Black diffusive

V=1

(V normalized to diffusive speed)

"The big picture"

•Significant progress has been made within the standard "V-D" diffusive paradigm.

•However, further progress seems to require a paradigm shift to relax the restrictive assumptions of the standard model.

•Non-local and no-equilibrium process play a key a role. We have presented a framework for the modeling of non-diffusive processes focusing in these two elements.

•The basic idea is to substitute the local Fick model by $q \sim \partial_x \int T(x',t)\, K(x-x')\, dx'$

•Different models have been proposed depending on the function K

•Our rationale for the election of K is based on the theory of non-Gaussian stochastic processes (continuum limit of generalized random walks).

•This approach brings together non-locality and non-Gaussian statistics resulting from non-equilibrium processes.

•In the case of α-stable Levy processes the models can be formulated using fractional derivative operators.

"The big picture"

- The connection with fractional calculus is very valuable from the analytical and numerical point of view.

- This has allowed the application of fractional diffusion models to transport problems in fusion plasmas.

- In particular, it has been shown that fractional diffusion models reproduce non-diffusive phenomenology including: anomalous scaling, up-hill transport, pinch effects, profile peaking, and fast cold pulse propagation phenomenal.

- Numerical and experimental perturbative transport studies provide a good guide for the construction and validation of non-diffusive transport models.

- In particular, the fractional model has been successfully applied to describe:
    - Tracers transport in pressure-gradient driven plasma turbulence
    - Edge cold pulse and power modulation experiments in JET

A lot of work remains to be done!!!!