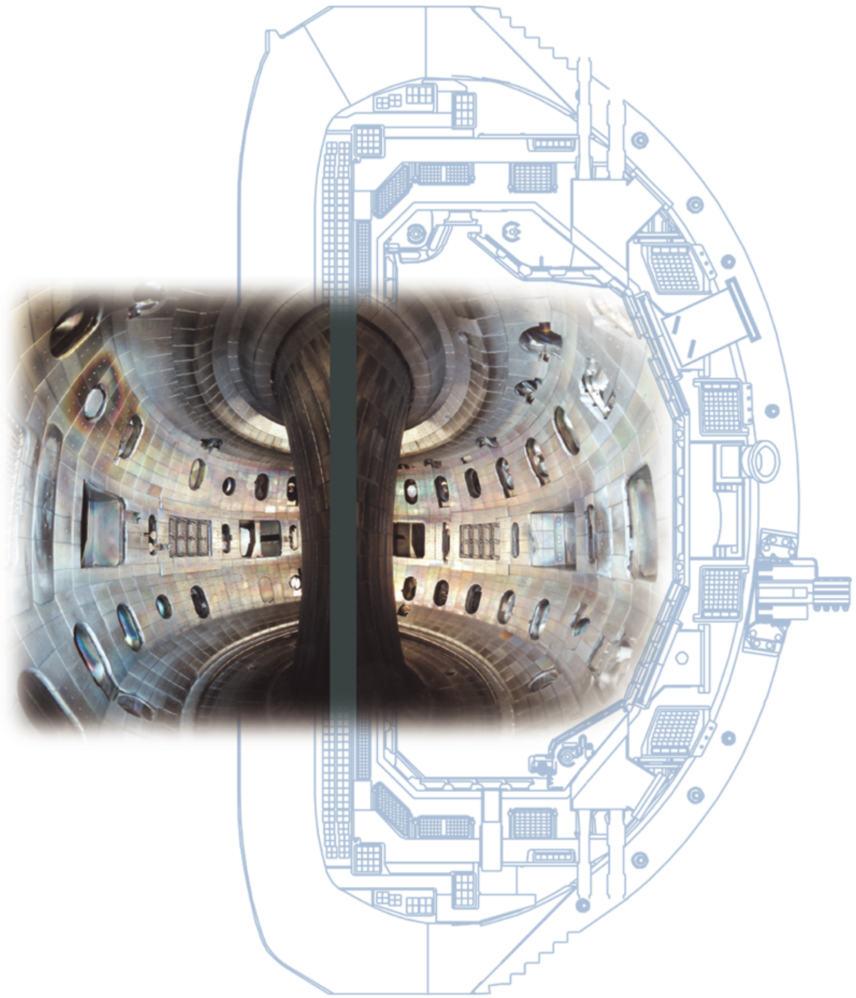
Introduction to the Science of Control

M.L. Walker
General Atomics, San Diego, USA



4th ITER International Summer School
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Objectives of Talk

- **Learn some control terminology**
- **Develop some intuition about control concepts**
  - Details occasionally (and intentionally) omitted
- **Understand the multiple objectives of control**

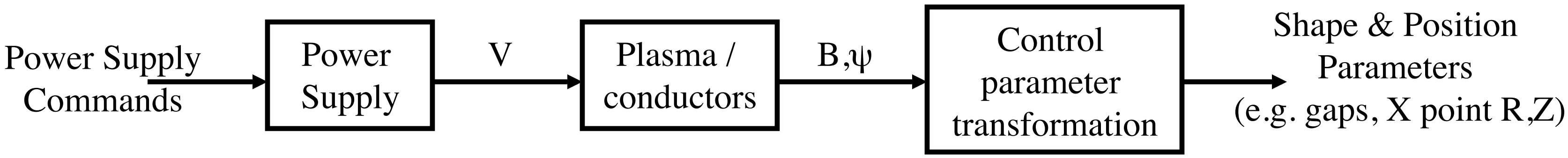
Model-Based Control Design Process

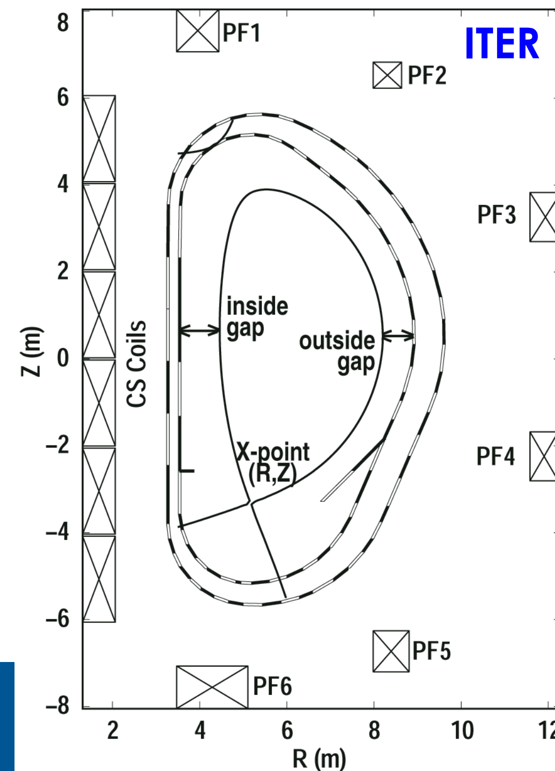
1. **Make system model**
2. **Verify model predicts behavior of system**
3. **Design controller**
4. **Test using models in closed-loop simulation**
5. **Implement and test implementation**
6. **Deploy in operation**

- **Using only 5-6 is feasible and often successful – why do steps 1-4?**
  - Requires empirical tuning, cost = $50,000 - $100,000 per day on present devices
  - Performance:
    - Large systems (many inputs / outputs) difficult to tune properly for best control
    - Nonlinear systems require retuning over many equilibrium states.
  - Even if Steps 5-6 is chosen approach, studying models is useful to understand how control affects system
  - Next Generation devices (e.g. ITER) will not allow empirical tuning

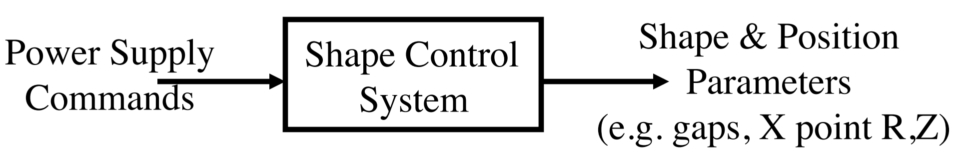
Introduction to System Representation - Block Diagrams

- **A Block Diagram consists of two parts:**
  - Signals (arrows in diagram)
  - Operations (blocks in diagram)
- **Example (poloidal field system producing plasma shape)**

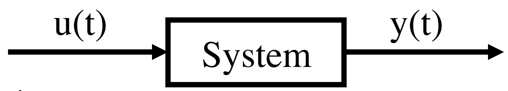




- **Equivalently, hiding all details:**



System Representation – Ordinary Differential Equations

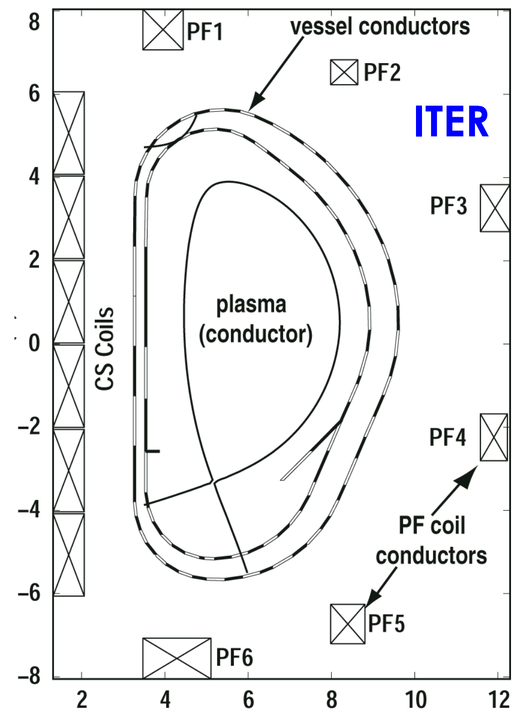


- **State Space Models**
  - General (x is "state"): $\dot{x} = f(x,u,t)$, $y = g(x,u,t)$
  - Note **ordinary differential equation (ODE)** is 1st order
  - Linear, time-invariant (LTI) system:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

  - Example (plasma + conductors):

$$M_* \frac{d(\delta I)}{dt} + R\delta I = U\delta v \qquad \Rightarrow \qquad d(\delta I)/dt = A\delta I + Bv$$
$$y = CI \qquad\qquad y = CI$$
$$(A = -M_*^{-1}R, \quad B = M_*^{-1}U)$$

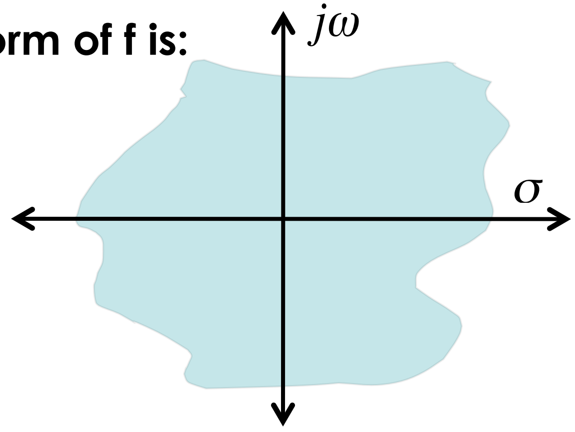


  - I(t) = toroidal conductor currents (perturbations δI from equilibrium ⇔ states x); M_*=mutual inductance matrix (modified by plasma response), R=resistance matrix
  - *y(t)* = coil currents, flux and field in vacuum region; C=green functions
  - *v(t)* = input voltage from power supplies (δv from equilibrium); U = ones for coils, zeros for vessel conductors

System Representation – Laplace Transform

- **Definition: For a given function f(t) with f(0)=0, Laplace transform of f is:**

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \qquad s = \sigma + j\omega$$



complex ("s") plane
("frequency domain")

- **Nice properties:**

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s), \quad \mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2 F(s), \quad \ldots etc \ldots$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

- **For an example of how it's used, apply to :**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

$$sX(s) = AX(s) + BU(s) \;\Rightarrow\; (s\mathbf{I} - A)X(s) = BU(s) \;\Rightarrow\; X(s) = (s\mathbf{I} - A)^{-1}BU(s)$$

$$\begin{aligned} Y(s) &= CX(s) + DU(s) \\ &= C(s\mathbf{I} - A)^{-1}BU(s) + DU(s) \;\Rightarrow\; Y(s) = \left(C(s\mathbf{I} - A)^{-1}B + D\right)U(s) \end{aligned}$$

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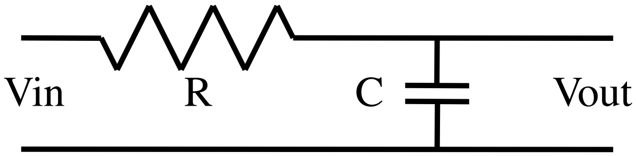
System Representation - Transfer Functions

- **Transfer Function = ratio of Laplace Transforms of (scalar) output and input signals:** $\frac{Y(s)}{U(s)}$
- **Example (simple mechanical system; x is displacement):**

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = u(t) \;\Rightarrow\; (ms^2 + ds + k)X(s) = U(s) \;\Rightarrow\; \frac{X(s)}{U(s)} = \frac{1}{(ms^2 + ds + k)}$$

- **Example (lowpass RC filter):**



$$\Rightarrow\; \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1}$$

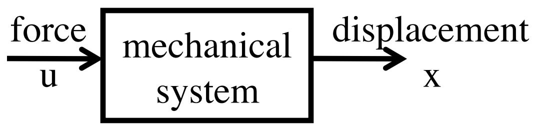
- **General LTI case, from previous page:** $Y(s) = \left(C(s\mathbf{I} - A)^{-1}B + D\right)U(s)$
- **If Y, U are scalars:** $\frac{Y(s)}{U(s)} = \left(C(s\mathbf{I} - A)^{-1}B + D\right)$ **(Single-Input-Single Output (SISO) system)**
- **If Multi-Input-Multi-Output (MIMO) system, each element in matrix** $C(s\mathbf{I} - A)^{-1}B + D$ **is a scalar transfer function, so still called "transfer function"**

System Representation - Equivalent Representations

- **Block Diagram**

State Space (1st order ODE)

Transfer Function

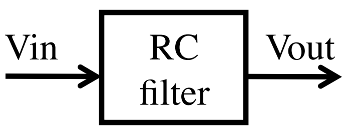


$$m\ddot{x} + d\dot{x} + kx = u$$

$$\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \frac{d}{dt}\begin{bmatrix} v \\ x \end{bmatrix} + \begin{bmatrix} d & k \\ -1 & 0 \end{bmatrix}\begin{bmatrix} v \\ x \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ x \end{bmatrix}$$

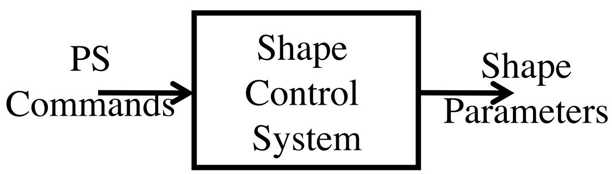
$$\frac{Y(s)}{U(s)} = \frac{1}{(ms^2 + ds + k)}$$



$$\dot{V}_{out}(t) = -\frac{1}{RC}V_{out}(t) + \frac{1}{RC}V_{in}(t)$$

$$y(t) = Vout(t)$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1}$$



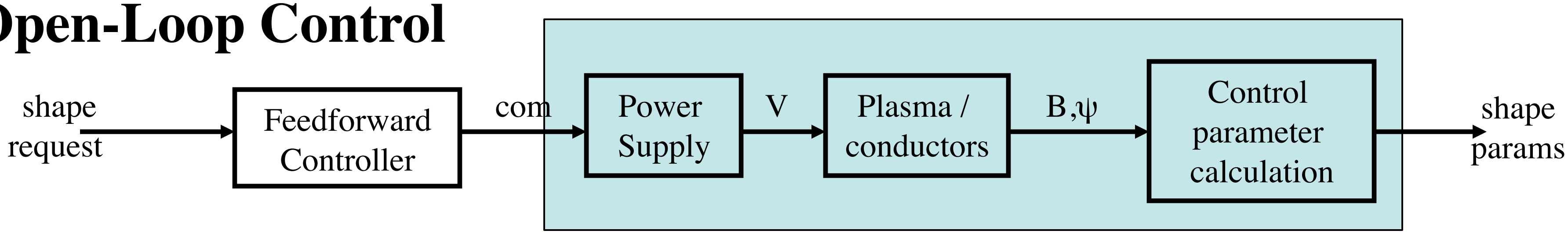
$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

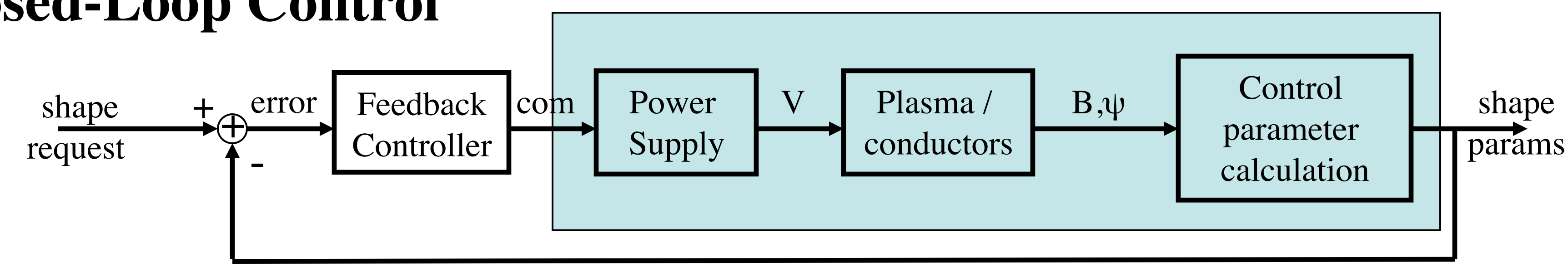
$$Y(s) = \left(C(s\mathbf{I} - A)^{-1}B + D\right)U(s)$$

System Representation – Feedforward/Feedback

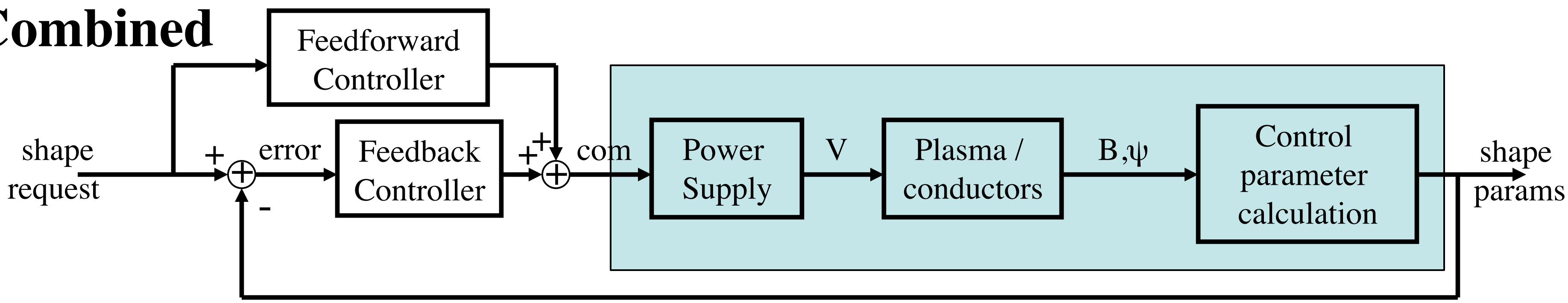
Open-Loop Control



Closed-Loop Control



Combined



Analysis of Dynamics (Time Dependent Behavior)

- **What is undriven "natural" behavior of system?**

$$\dot{x}(t) = Ax(t) + \cancel{Bu}(t)$$

- **Defined by the eigenvalues λ:**

$$\lambda x = Ax$$

- **An arbitrary vector v can be expressed as sum of eigenvectors:** $v = \sum_{k=1}^{n} \alpha_k x_k$

- **Then:** $Av = \sum_{k=1}^{n} \alpha_k A x_k = \sum_{k=1}^{n} \alpha_k \lambda_k x_k \quad \Rightarrow \quad \dot{x} = \sum_{k=1}^{n} \alpha_k \dot{x}_k = \sum_{k=1}^{n} \alpha_k \lambda_k x_k$

- **That is, we can analyze as n scalar ODE's:**

$$\dot{x}_k = \lambda_k x_k \quad \Rightarrow x_k(t) = e^{\lambda_k t} x_k(0)$$

- **To determine stability of the system:**

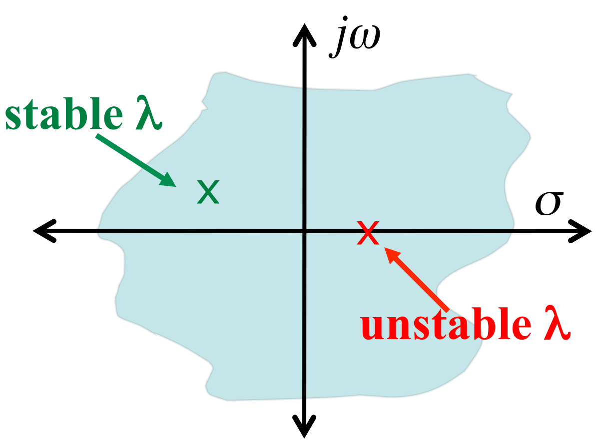
$$\sigma_k = real(\lambda_k) < 0 \quad \Rightarrow \quad x_k(t) \to 0,\ t \to \infty \qquad \text{(stable)}$$

$$\sigma_k = real(\lambda_k) > 0 \quad \Rightarrow \quad x_k(t) \to \infty,\ t \to \infty \qquad \text{(unstable)}$$

- **If ANY eigenvalue has Re(λ)>0 => system is UNSTABLE.**
- **Otherwise, system is STABLE.**

complex-plane

$(\lambda = \sigma + j\omega)$

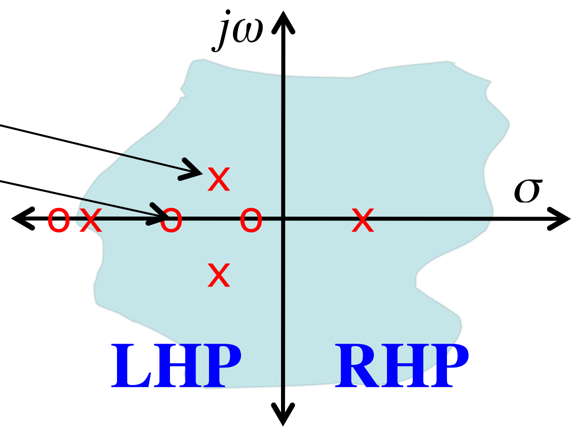


Analysis of Dynamics (Laplace Domain)

- **Poles and zeros of Transfer Functions:** $\frac{y(s)}{u(s)} = \frac{b(s)}{a(s)}$ ← polynomials
  - Complex function theory terminology:
    - Roots of denominator polynomial a(s) = **poles**
    - Roots of numerator polynomial b(s)= **zeros**

complex-plane

$(s = \sigma + j\omega)$



- **If ANY poles have σ=Re(s)>0, system is UNSTABLE,**
- **otherwise, STABLE. (Explanation in a moment.)**
- **Examples:**

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs+1}$$

has 1 poles (in LHP) and no zeros => STABLE

$$\frac{Y(s)}{U(s)} = \frac{1}{(ms^2 + ds + k)}$$

has 2 poles (in LHP) and no zeros => STABLE

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{RCs}{RCs+1}$$

(high-pass filter) has 1 pole (in LHP) and 1 zero (at 0) => STABLE

LHP/RHP = Left/Right Half Plane

Analysis of Dynamics (Time vs. Laplace Domains)

- **Eigenvalue is a complex number λ satisfying:**
  - $(\lambda I - A)x = 0$ for some $x \neq 0$
  - $\Leftrightarrow$ $(\lambda I - A)^{-1}$ does not exist
  - $\Leftrightarrow$ determinant $|\lambda I - A| = 0$
- **Note similarity to portion of Transfer Function:**

$$Y(s) = \left(C(s\mathbf{I} - A)^{-1}B + D\right)U(s)$$

- **In fact,**

$$(s\mathbf{I} - A)^{-1} = \frac{1}{|s\mathbf{I} - A|} Adj(s\mathbf{I} - A)$$

where:

$|X| =$ **determinant of** X

$Adj(X) =$ **adjugate of** X **(matrix of cofactors)**

- **A common situation is D=0, so that the transfer function is:**

$$\frac{1}{|s\mathbf{I} - A|}\left(CAdj(s\mathbf{I} - A)B\right)$$

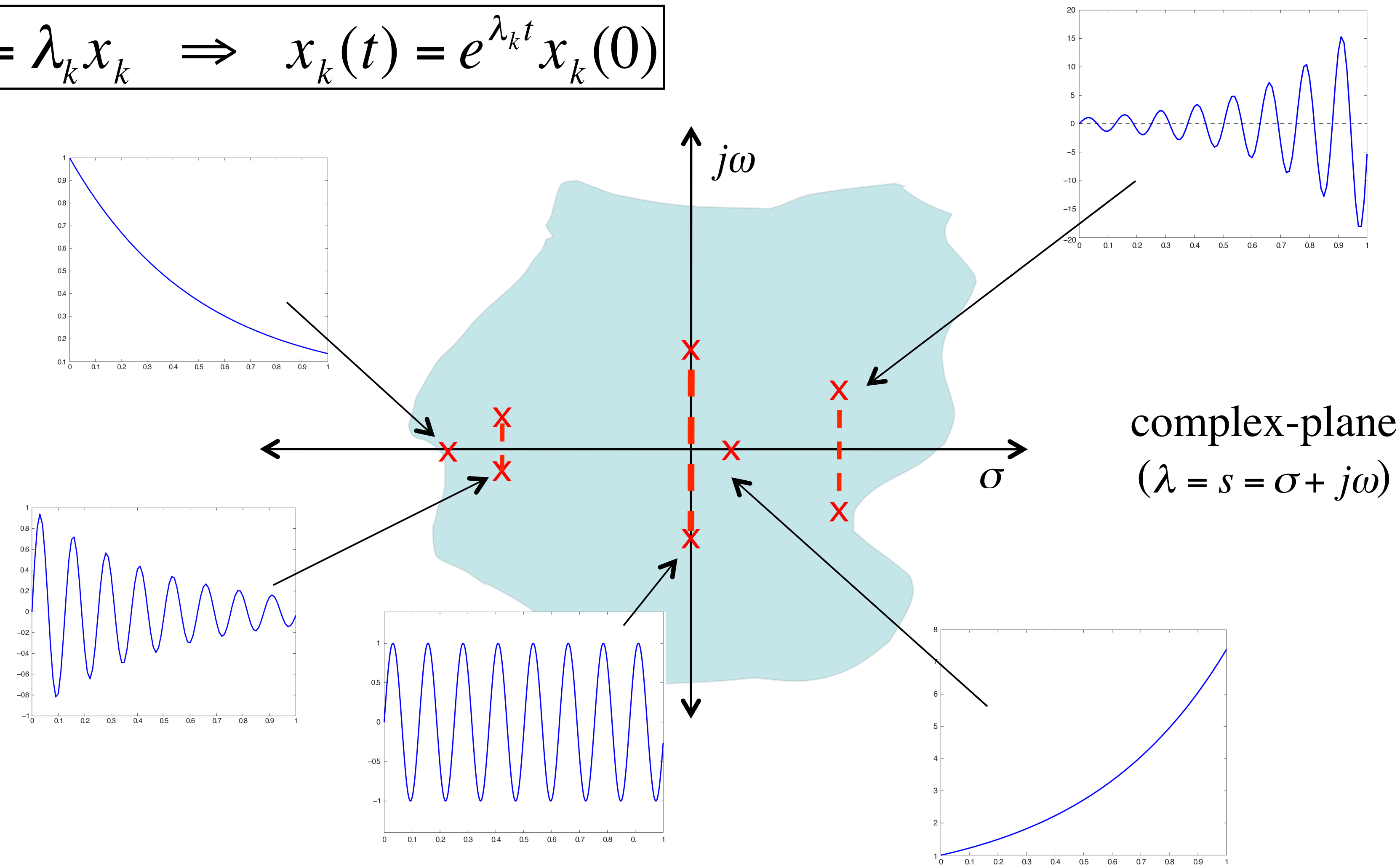
polynomial in s (denominator $|s\mathbf{I} - A|$)

matrix of polynomials in s ($CAdj(s\mathbf{I} - A)B$)

- **That is, the POLES of the transfer function = roots of determinant of (sI-A)**
  = EIGENVALUES of A

Understanding System Response – Correspondence Between Eigenvalue (Pole) Location and Time Response

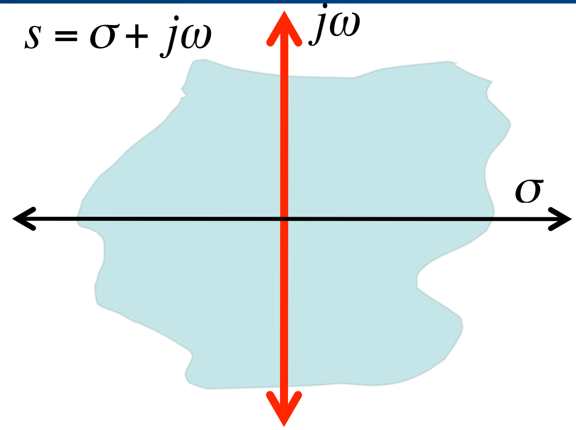
$$\dot{x}_k = \lambda_k x_k \quad \Rightarrow \quad x_k(t) = e^{\lambda_k t} x_k(0)$$



Understanding System Response – Frequency Response

- **Recall Laplace Transform definition:**

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$



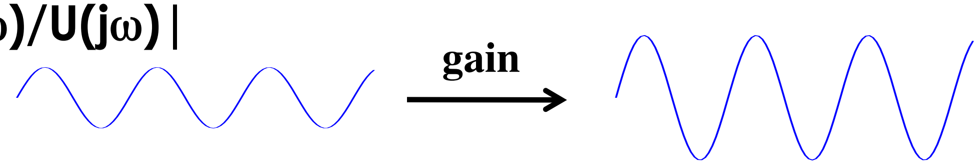
- **Restrict to jω axis obtains Fourier Transform if f(t<0)=0 :**

$$F(j\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^\infty e^{-j\omega t} f(t)dt$$

- **For a system with transfer function Y(s)/U(s),**

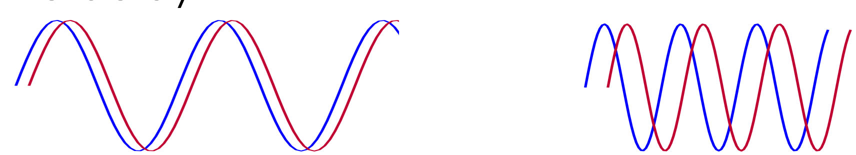
$$\frac{Y(j\omega)}{U(j\omega)} = \frac{|Y(j\omega)|e^{j\cdot phase(Y(j\omega))}}{|U(j\omega)|e^{j\cdot phase(U(j\omega))}}$$

- **System Gain is defined to be |Y(jω)/U(jω)|**



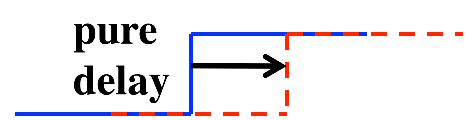
- **System Delays: Two types:**
  - Phase lag = frequency dependent time delay

$$lag = phase(Y(j\omega)) - phase(U(j\omega))$$



low frequency = small delay, high frequency = large delay

  - Pure delay = frequency independent time delay



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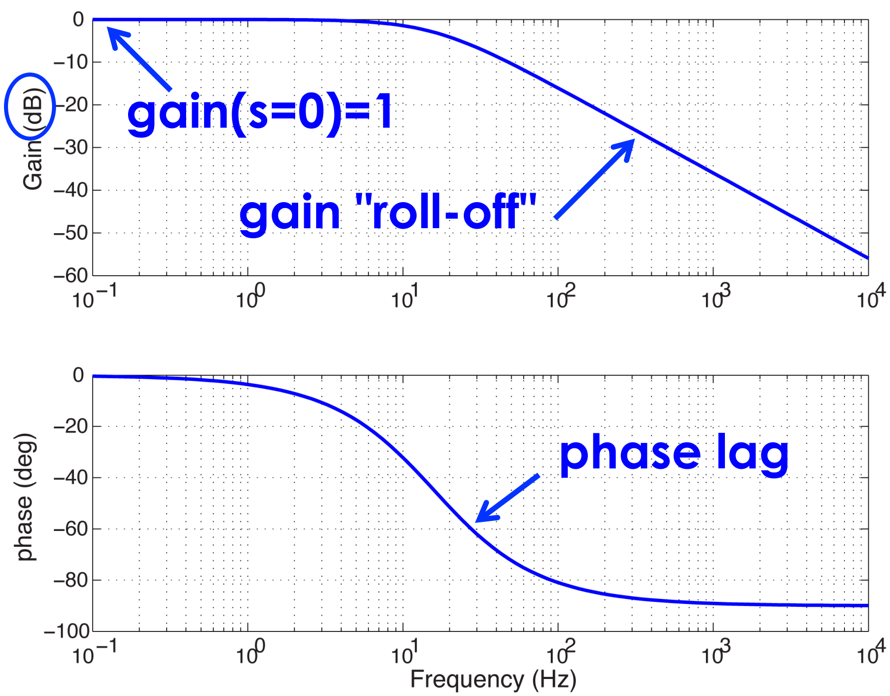
Understanding System Response – Bode Plots of Frequency Response

- **Examples:**

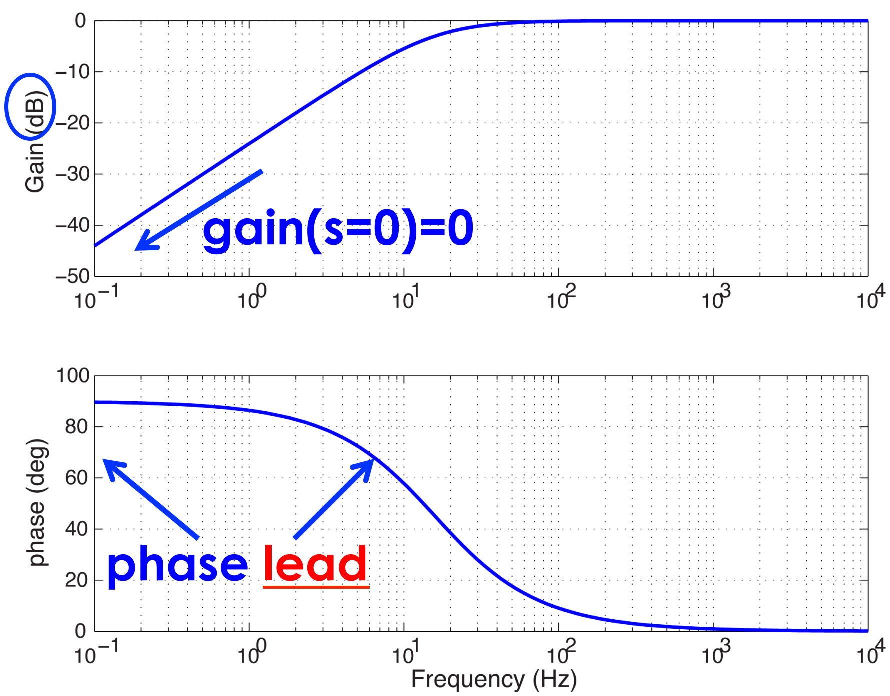
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs+1}$$

Lowpass filter



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{RCs}{RCs+1}$$

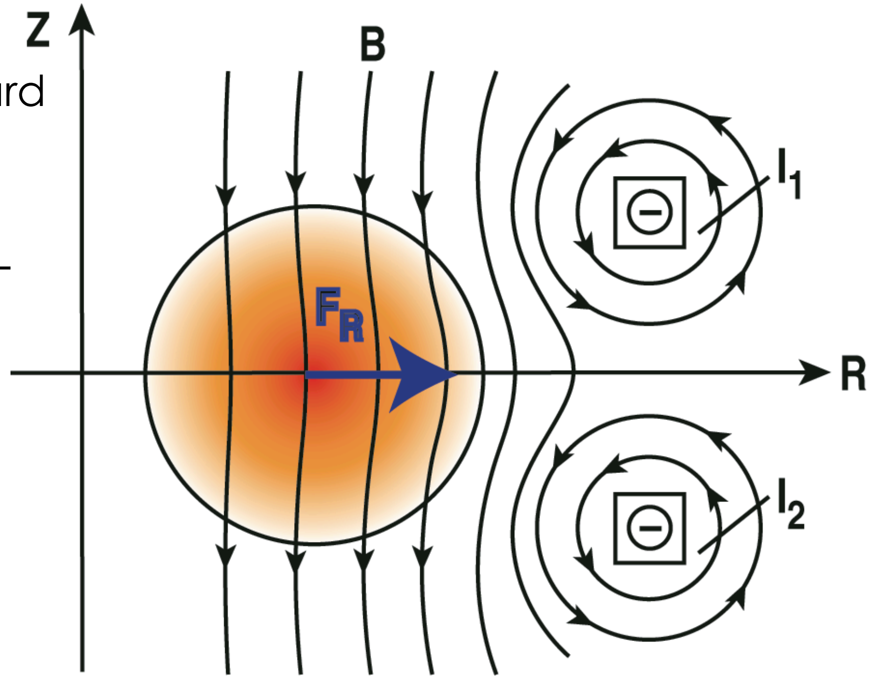
Highpass filter

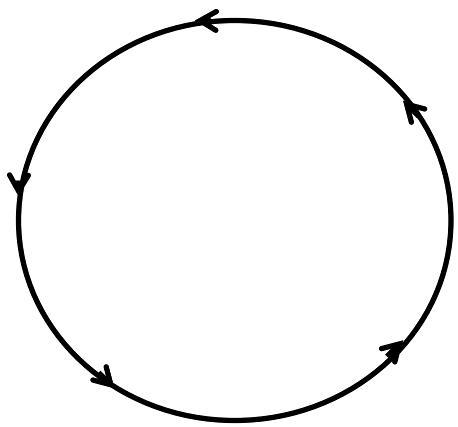


NOTE: Bode gain plot is ratio of <u>powers</u> ($20\log_{10}$(amplitude ratio)).

Objectives of Control – Tracking and Regulation

- **Control plasma major radius:**
  - Assume plasma current (I_p) is positive
  - Radial hoop force F_R pushes plasma outward
  - Vertical field (B_Z) produced by outer coils holds it in desired location (**regulation**) ...
  - ... or moves plasma in/out to match a time-dependent request (**tracking**)





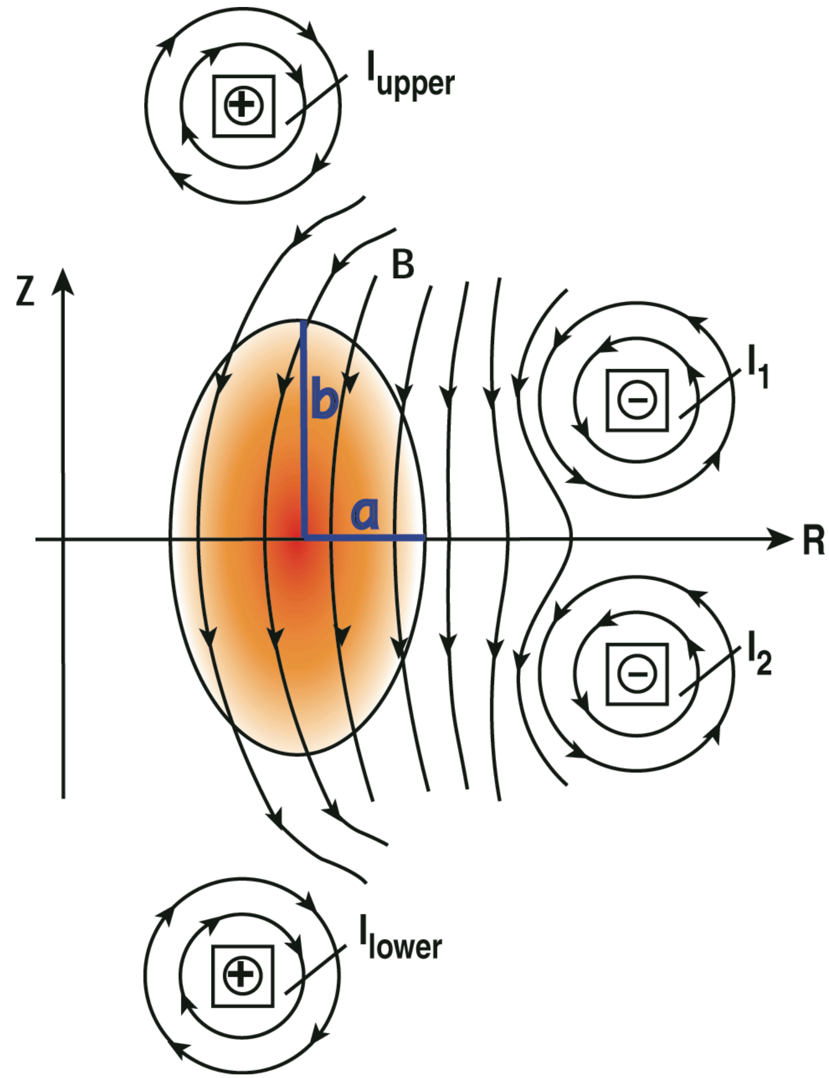
tokamak positive current sign convention (viewed from above)

Objectives of Control – Tracking and Regulation

- **Control plasma elongation:**
  - Increasing elongation (κ) has been shown to improve performance, so we want to control:

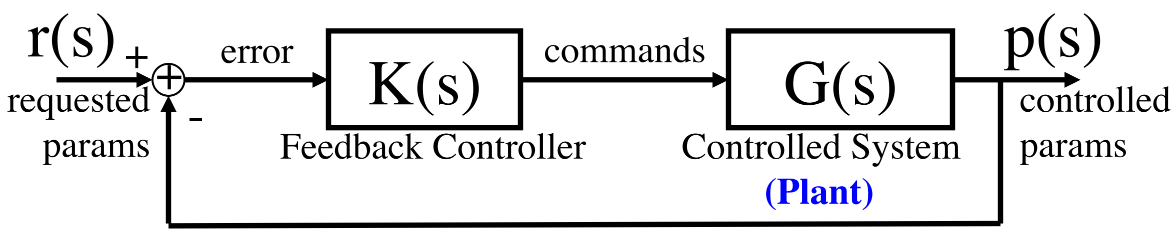
$$\kappa = \frac{b}{a}$$

  - Control accomplished by "pulling" on top and bottom of plasma
  - However, elongating plasma introduces destabilizing field curvature (explained in a moment)



Objectives of Control – Tracking and Regulation

- **Derivation of Closed-Loop Transfer Function:**



$$p(s) = G(s)K(s)(r(s) - p(s))$$

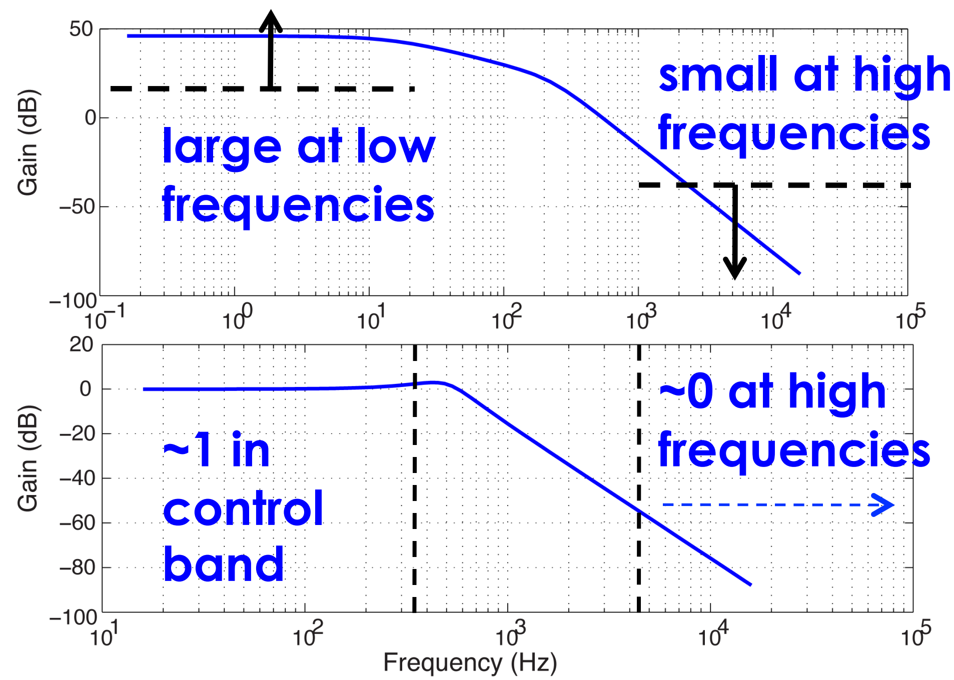
$$(I + G(s)K(s))\,p(s) = G(s)K(s)r(s)$$

$$\Rightarrow \frac{p(s)}{r(s)} = \frac{G(s)K(s)}{(I + G(s)K(s))}$$

- **What we want:**

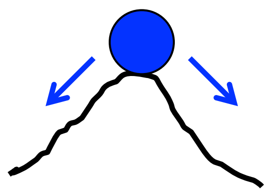
Open-Loop Transfer Function $G(s)K(s)$

Closed-Loop Transfer Function $\frac{G(s)K(s)}{(I + G(s)K(s))}$

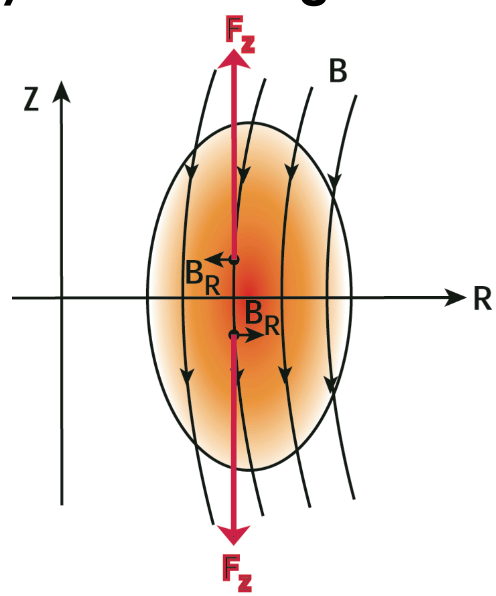


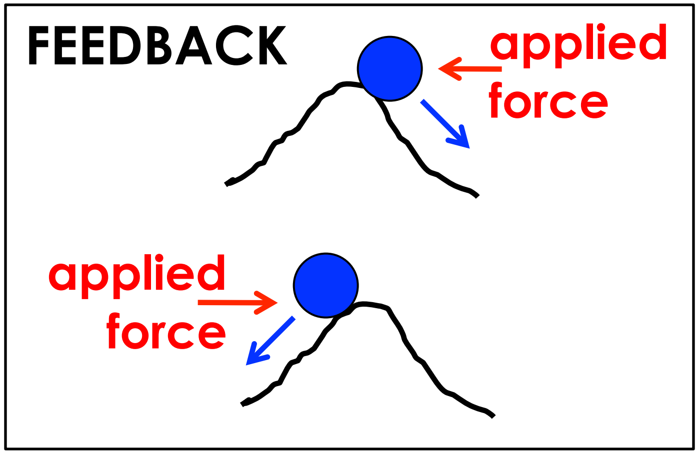
Objectives of Control - Stabilization

- **Open-loop instability:**

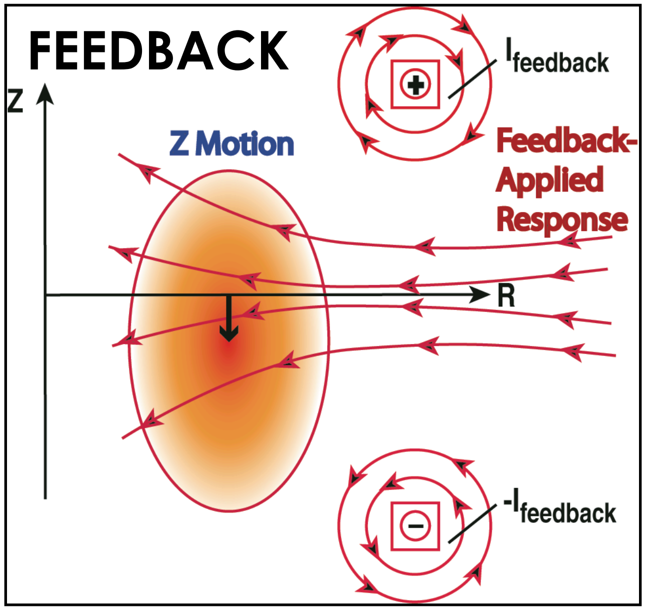


- **Plasma vertical instability (caused by destabilizing curvature):**



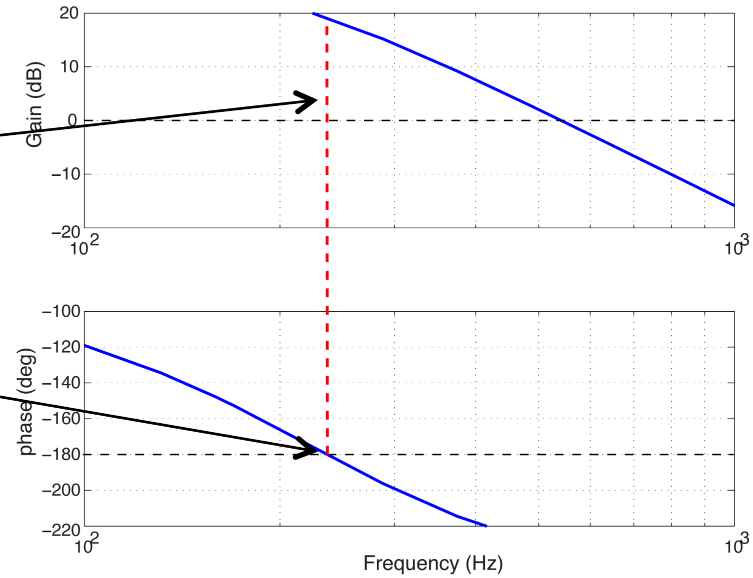


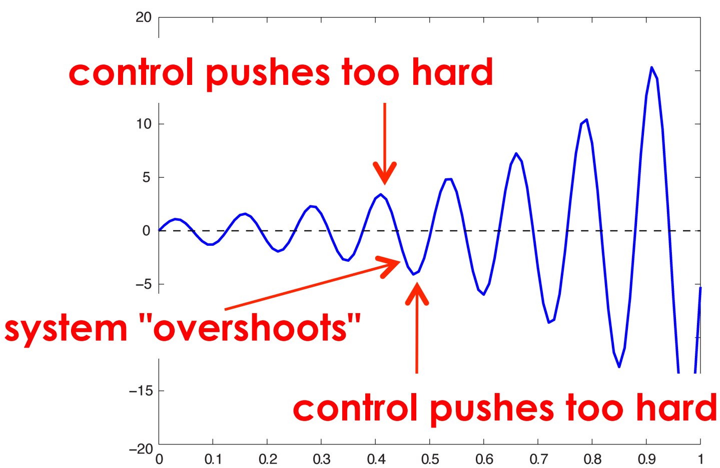
Anti-symmetric coils provide radial field to apply force that opposes plasma vertical motion



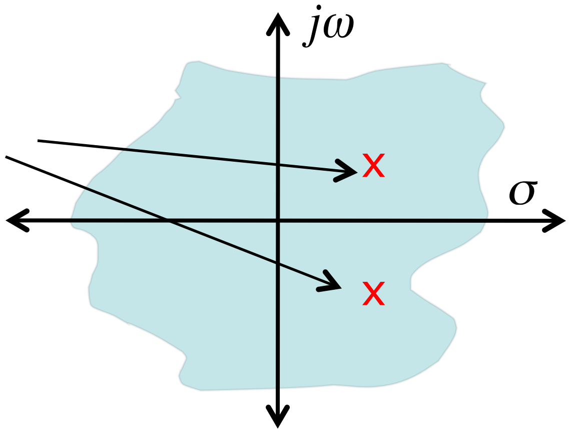
Objectives of Control – Avoid Closed Loop Instability

- Gain cannot be considered independently from phase.
- If gain > 1 ....
- ... when phase = -180 (opposite sign)
- => positive-feedback at that frequency and result is control-driven instability...





... and closed-loop transfer function has pair of poles in RHP



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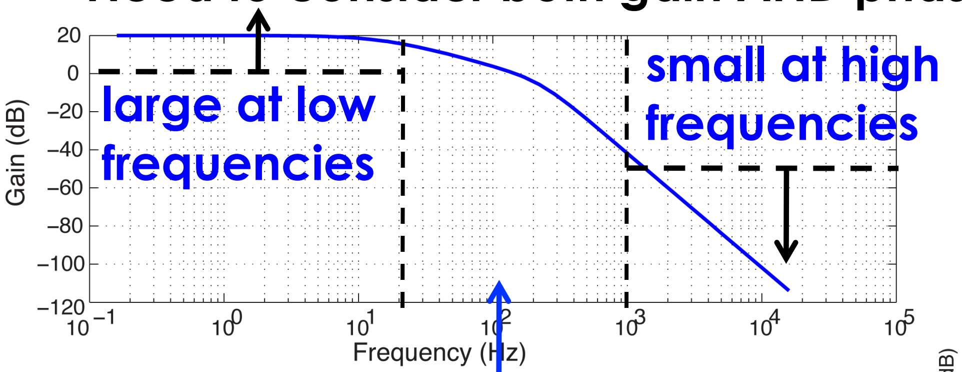
Objectives of Control – Closed Loop Stability

- **Need to consider both gain AND phase:**

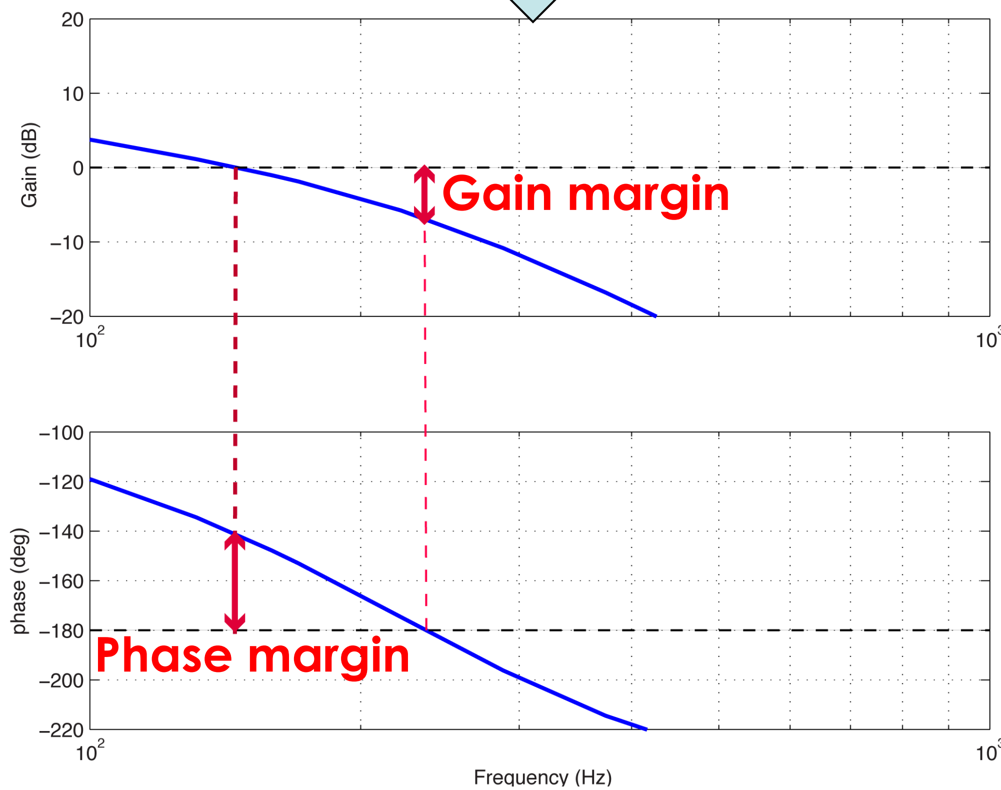


Previous gain K reduced to achieve positive gain and phase margins.

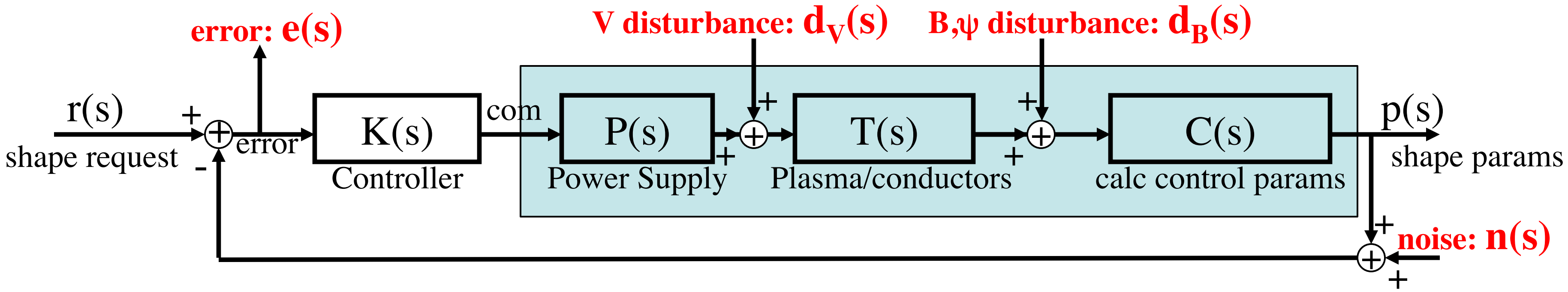
Pay attention to stability (phase) in the middle

- Need gain <<1 for phase=-180°
- Need phase lag << 180° for gain >1

(Gain/Phase margins are one example of stability margins.)



Objectives of Control – Disturbance & Noise Rejection



- **Disturbance rejection means ratio of norms of errors to input is small:**

$$\frac{\|e(s)\|}{\|d_V(s)\|} << 1, \quad \frac{\|e(s)\|}{\|d_B(s)\|} << 1$$

(attenuate effect of disturbances)

- **Noise rejection means ratio of norms of errors to input noise is small:**

$$\frac{\|e(s)\|}{\|n(s)\|} << 1$$

(attenuate effect of noise)

- **These are ensured by making norms of transfer functions small, e.g.:**

$$\frac{\|e(s)\|}{\|d_V(s)\|} \leq \left\| -(I + CTPK)^{-1} CT \right\| << 1$$

- **For example, large gains in controller K can make this small.**

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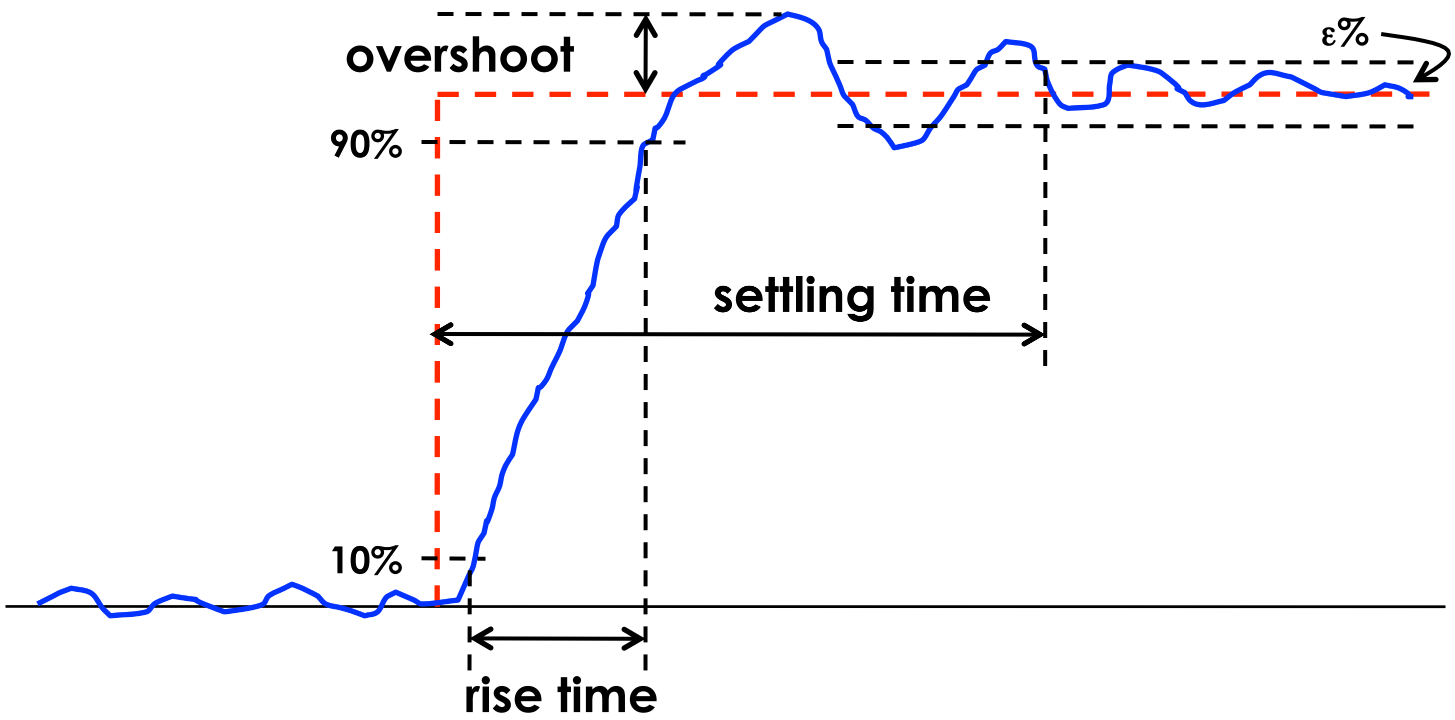
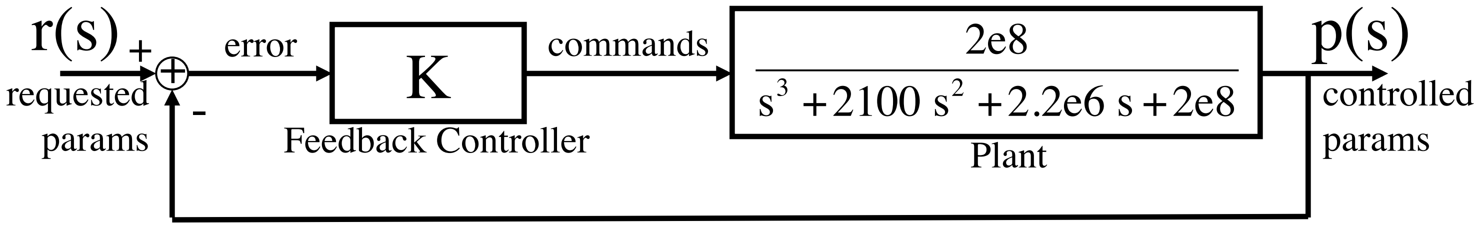
Performance Requirements – Time Domain

- **Typical Specifications on Step Response:**
  - Rise Time < X seconds
  - Percent Overshoot < Y %
  - Settling Time < Z seconds (within ε %)

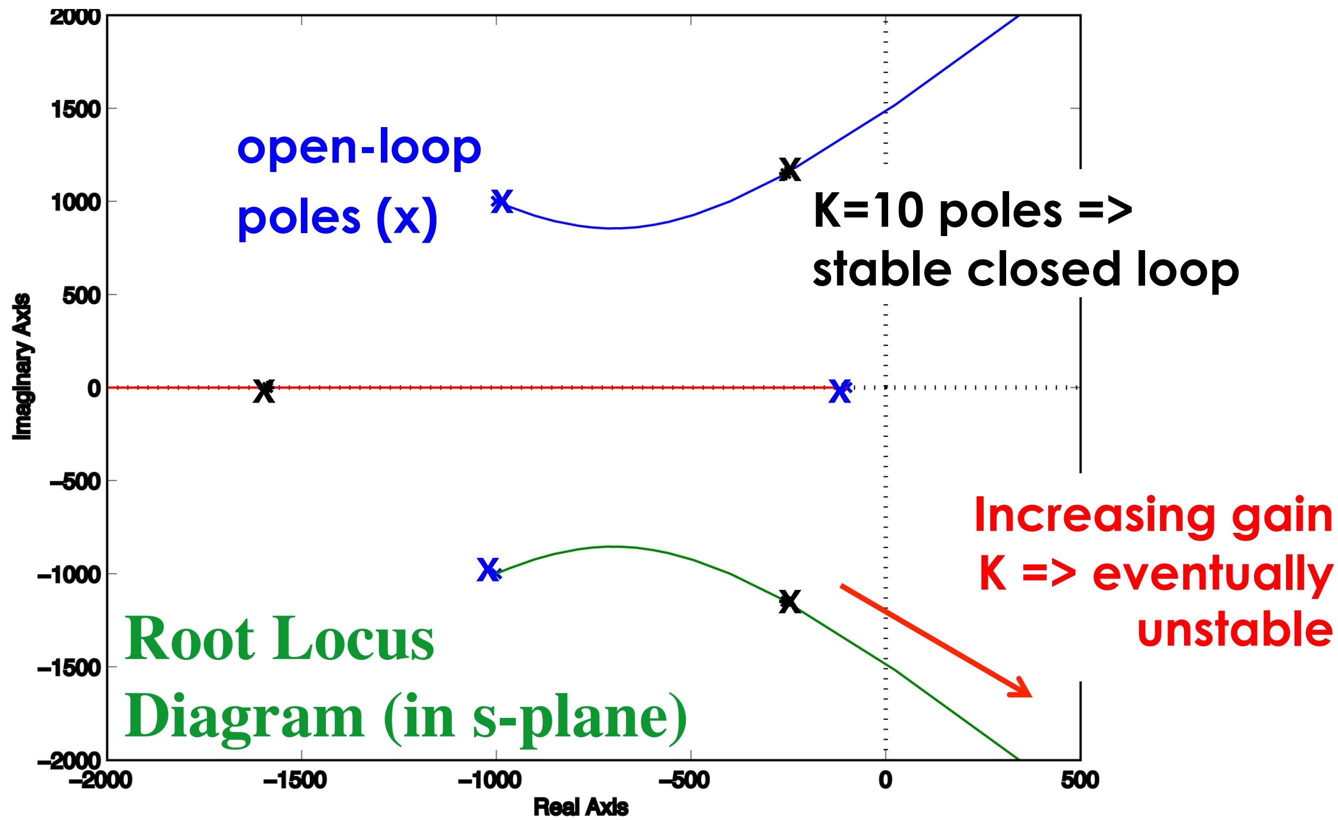


Performance Requirements - Stability

- **Consider plant used in Bode plots:**

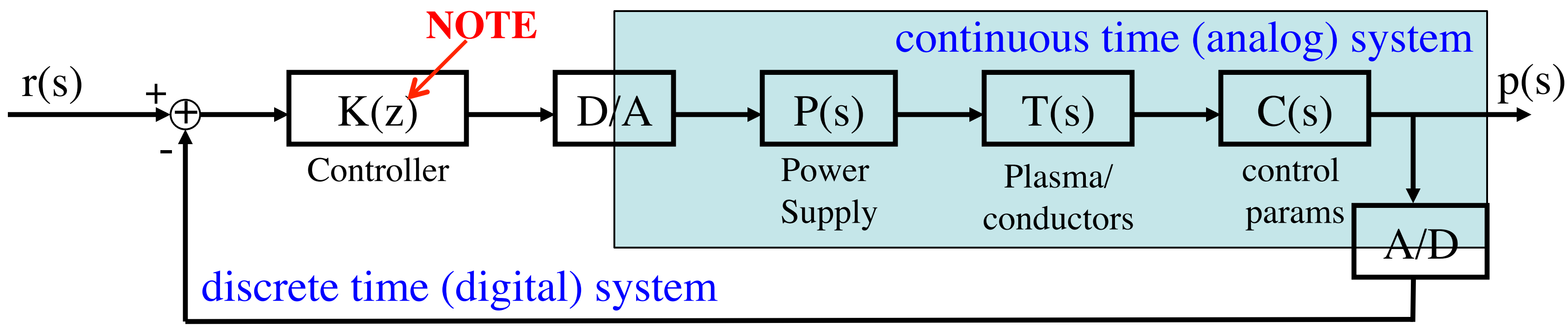


- **Root Locus diagram shows stability changes with K:**
  - Open-loop stable plant
  - Stable closed loop, K=10
  - Unstable closed loop, K=200



System Representation – Sampled Data Systems

- **Modern plasma control mixes discrete- and continuous-time systems:**



- **Approach (1) to Control Design:**
  - Treat entire system as continuous time. Develop continuous controller K(s), then convert to discrete controller K(z).
  - Issues: Close to original physics models, but sampling rate must be fast enough to justify treating discrete controller as continuous.
- **Approach (2) to Control Design:**
  - Treat entire system as discrete and develop discrete controller directly. (Methods exist to convert mixed continous/discrete to all discrete system.)
  - Issues: Direct production of discrete controller with given sample rate, but difficult to retain physical intuition.

System Representation – Discrete Time Systems

- Time now represented by integers k=1,2,… (i.e., time = sample number)

- State-Space models are difference equations:

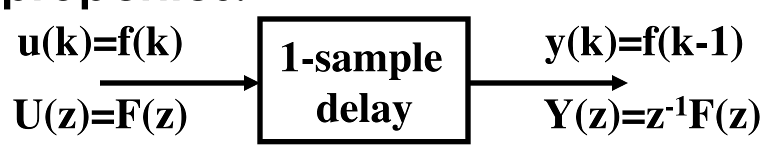
$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

- Now we have Z-transform instead of Laplace transform

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

- Nice properties:

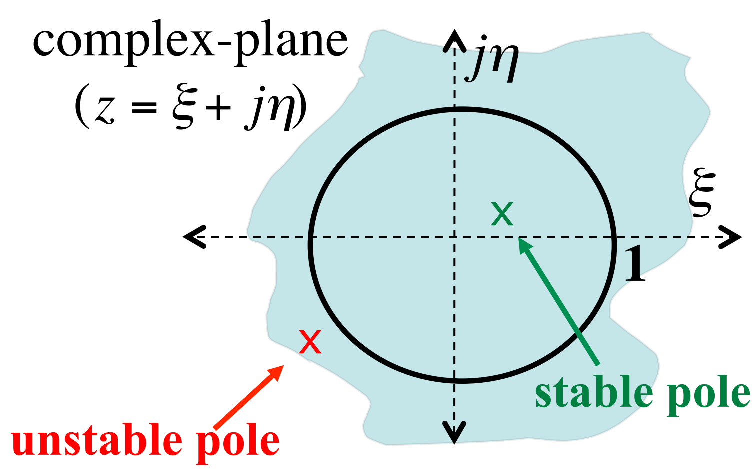


- Transfer functions now defined on "z"-plane:

$$zX(z) = AX(z) + BU(z)$$

$$Y(z) = CX(z) + DU(z)$$

$$\Rightarrow Y(z) = C\left((zI - A)^{-1} BU(z)\right) + DU(z)$$



Controllers – Example Digital Implementations

- **Simple gain multiplier:**
  - Command signal u(k) = K * e(k)   (error e(k) = r(k) - y(k))
  - K can be scalar (SISO) or matrix (MIMO)
- **Digital filter (SISO):** only previous samples

$$u(k) = a_1 u(k-1) + \ldots + a_n u(k-n) + b_0 e(k) + b_1 e(k-1) + \ldots + b_m e(k-m) \Rightarrow \frac{U(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 - a_1 z^{-1} - \ldots - a_n z^{-n}}$$

present and previous samples

- **State Space:**
  - Either SISO or MIMO:

$$u(k) = C_c x_c(k-1) + D_c e(k)$$
$$x_c(k) = A_c x_c(k-1) + B_c e(k)$$

  - Output computed from present error and previous state
  - Controller state is updated at each time step

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Next – some examples of types of controllers

- **Why different controller types?**
  - Simple versus difficult to use
  - SISO versus MIMO system
  - Highly coupled versus mostly diagonal system
  - How problem is posed (what you "care about")
  - Noise characteristics of system
  - Disturbance sources/effects and characteristics
  - Level of knowledge of system dynamics (model uncertainty)
  - Guaranteed stability including uncertainty versus nominal stability (not accounting for uncertainty)
  - Guaranteed performance including uncertainty versus nominal performance (not accounting for uncertainty)

Controller Types – PID controllers

- **PID = Proportional, Derivative, Integral feedback**
  - Ideal: $u(t) = K_P e(t) + K_D \dot{e}(t) + K_I \int e(t)dt$
  - e(t) = error signal, u(t) = command to control actuator
- **Simple and often all that is needed (DO NOT confuse "often" with "always")**
- **Purpose of each term:**
  - K_P: Tracking ($K_P G/(1+K_P G) \sim 1$ over control bandwidth)
  - K_I: Regulation (gain is infinite at $j\omega=0$ => steady-state error = 0)
  - K_D: Damping, phase lead
- **Issues:**
  - K_P: can destabilize if too large (implemented as simple gain multiplier)
  - K_I: integrator windup (implemented as digital filter)
  - K_D: amplifies noise at high frequencies (implemented as digital filter)
- **Advantage:**
  - Simple, tunable
- **Disadvantage**
  - Difficult to determine gains in highly coupled systems

Controller Types – LQG controllers

- **LQG= Linear, Quadratic, Gaussian ("optimal control")**
  - Assume the **linear** system has **Gaussian** noise v(t), w(t):
    $$\dot{x}(t) = Ax(t) + Bu(t) + v(t)$$
    $$y(t) = Cx(t) + w(t)$$
  - Minimize objective functional J ...
  - ... where Q>0, R>0 (**quadratic** cost)
    $$J = \int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)dt$$
  - Typically, states x are variations around a **stable** equilibrium x_0
  - Sometimes J has terms for output y or error e = reference - output
- **Main idea: keep signals small "on average" (variation due to noise)**
- **Optimal controller is given by:**
  $$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$
  $$u(t) = -L\hat{x}(t)$$
  - First equation is the **Kalman Filter**, which provides an **optimal** estimate for x
  - If state measured directly, insert x in place of x-hat and use 2^{nd} equation only
- **Advantage:**
  - Straightforward to generate controller optimal against "noise", once J is defined
- **Disadvantage**
  - Matrices Q and R typically determined through trial and error

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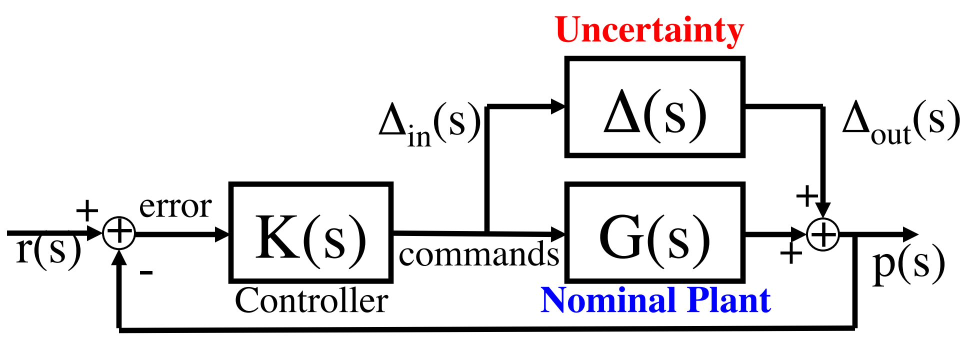
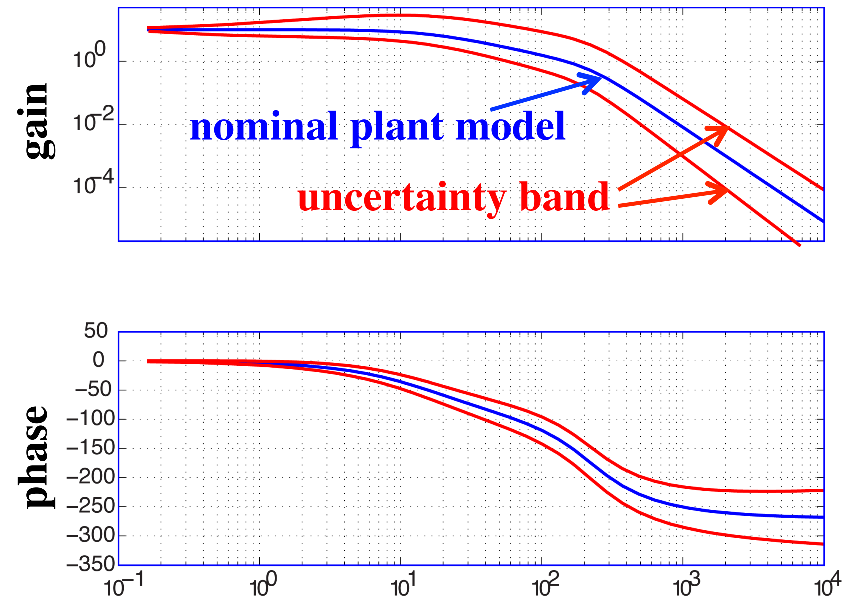
Controller Types – H-infinity ("robust") controllers

- $\mathbf{H}^{\infty}$= **method for synthesizing robust controllers ("Hardy space, infinity norm")**
- **Robust = guaranteed stability/performance with unknown (but bounded) uncertainty in plant model**
  - Infinity ("worst case") norm : $\|\Delta\|_{\infty} < bound$





- **Main idea**
  - Remove Δ from picture ...
  - ... and make transfer function from Δ_{out} to Δ_{in} **as small as possible**
- **Advantage:**
  - Guarantees on stability and performance in the deployed feedback system
- **Disadvantage:**
  - More difficult to understand and to use; some tools produce conservative designs

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Summary

- **Control Terminology and Concepts:**
  - Linear/Nonlinear systems, Linear-Time-Invariant system, Discrete time system, System gain/phase, s-plane, z-plane, poles, zeros, pure delay, phase lag, phase lead, SISO, MIMO, feedforward, feedback, open-loop instability, control-driven instability, LHP, RHP, frequency response, roll-off, gain margin, phase margin, stability margin, disturbance, overshoot, rise time, settling time
- **Control Tools and Methods:**
  - Block Diagrams, Transfer Functions, State Space Models, Laplace Transform, Z-Transform, Fourier Transform, Bode plot, derivation of closed-loop transfer function, Root Locus, PID controllers, LQG controllers, H-infinity controllers
- **Multiple Objectives of Control:**
  - Stability,
  - Tracking and Regulation
  - Disturbance Rejection
  - Noise Rejection
  - Robustness

Further Reading

- **Free downloadable books:**
  - Wikibook of automatic control systems, http://en.wikibooks.org/wiki/Control_Systems (not how you would want to learn control, but useful as a reference)
  - Kwaakernak and Sivan, Linear Optimal Control Systems, http://www.ieeecss.org/PAB/classics/
  - Wikibook of signals and systems, http://en.wikibooks.org/wiki/Signals_and_Systems
  - Matlab documentation at http://www.mathworks.com/access/helpdesk/help/helpdesk.html
    - Control System Toolbox, Robust Control Toolbox
- **Good entry-level control books:**
  - Franklin, Powell, Emami-Naeini, Feedback Control of Dynamic Systems
  - Friedland, Control System Design: An Introduction to State-Space Methods